EXTREMAL PROBLEMS FOR FUNCTIONS OF POSITIVE REAL PART AND APPLICATIONS

by

Vo Van Anh, B.Sc.(Hons.)

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Except as stated herein, this thesis contains no material which has been accepted for the award of any other degree or diploma in any University, and that, to the best of my knowledge and belief this thesis contains no copy or paraphrase previously published or written by another person, except when due reference is made in the text.

V.V. Anh.

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to L.T.H.

ABSTRACT

Let $\mathcal{B}$ be the class of functions $w(z)$ regular in $|z| < 1$ and satisfying $w(0) = 0$, $|w(z)| < 1$ in $|z| < 1$. We denote by $P(A, B)$, $-1 \leq B < A \leq 1$, the class of functions $p(z) = 1+p_1z+\dots$ regular in $|z| < 1$ and such that $p(z)=[1+Aw(z)]/[1+Bw(z)]$ for some $w(z) \in \mathcal{B}$. This thesis is concerned with establishing bounds on $|z|=r<1$ for functionals of the form

$$\mathrm{Re}\{\alpha p(z) + \beta zp'(z)/p(z)\} \quad , \quad \alpha,\beta \text{ real} ,$$

where $p(z)$ varies in $P(A, B)$ or one of the following subclasses:

$$P_k(A, B) = \{p(z) = 1+p_kz^k+p_{2k}z^{2k}+\dots \in P(A, B), \ k = 1,2,3,\dots\} \quad ,$$

$$P_b(A, B) = \{p(z) \in P(A, B) \ ; \ p'(0) = b(A-B) \ , \ 0 \leq b \leq 1\} \ ,$$

$$P[a, b] = \{p(z) \in P \equiv P(1,-1) \ ; \ p(a)=b \ , \ 0 < a < 1 \ , \ b > 0\} \quad .$$

The bounds obtained are used to derive the distortion theorems, the covering theorems and the radii of convexity for the classes of regular or meromorphic starlike functions associated with $P(A, B)$ or the above-mentioned subclasses.

Furthermore, we obtain bounds for the functional $\mathrm{Re}\{p(z)^{\alpha}-\alpha zp'(z)/p(z)\}$, $0 < \alpha \leq 1$, $p(z) \in P$, and establish the above theorems for the class of meromorphic strongly starlike functions of order α.

The problem of minimising the functional $\mathrm{Re}\{zp'(z)/p(z)\}$ over $P(A, B)$ is also examined for the case in which we may have $A \geq 1$. This situation arises from the investigation of the starlikeness of functions $f(z)$ normalised, regular in $|z| < 1$ and satisfying $|f(z)/[\lambda f(z)+(1-\lambda)g(z)]-\gamma|<\gamma$, $\gamma \geq 1$, $0 \leq \lambda < 1$, where $g(z)$ belongs to, for example, the class S^*_α of starlike functions of order α .

Finally we investigate the β-convexity of certain subclasses of starlike functions. In particular, the radii of β-convexity, β real, for the class S^*_α , $0 \leq \alpha \leq \frac{1}{2}$, and the class

$$S^*[\alpha] = \{f(z) = z+a_2z^2+\dots;|zf'(z)/f(z)-1|/|zf'(z)/f(z)+1|<\alpha, \ 0<\alpha\leq 1, \ z\in\Delta\} ,$$

are completely determined .

Several results of Chapter 1 are to appear in *J. Math. Anal. Appl.* (see [92]). Most of the material of Chapter 6 has been published in *Pacific J. Math.* (see [4]). Results of Chapter 7 have been submitted for publication (see [5]).

CONTENTS

CHAPTER 0

INTRODUCTION AND PRELIMINARIES

0.1 Introduction

A function $f(z)$ which is regular or meromorphic in a domain D of the complex plane is said to be *univalent* in D if for z_1, z_2 in D, $z_1 \neq z_2$, we have $f(z_1) \neq f(z_2)$. Throughout the thesis, unless otherwise stated, we shall confine ourselves to the case in which D is the unit disc $\Delta = \{z; |z| < 1\}$.

We denote by $\mathcal{S}$ the family of functions $f(z)$ regular and univalent in Δ with series expansion

$$(0.1.1) \qquad f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

and $\sum$ the family of functions $g(z)$ meromorphic and univalent in Δ with Laurent expansion

$$(0.1.2) \qquad g(z) = \frac{1}{z} + b_0 + b_1 z + \dots$$

The class of regular functions $f(z)$ with the normalisation $f(0) = 0$, $f'(0) = 1$ as given by (0.1.1) is denoted by $\mathcal{N}$.

In attempting to verify the Bieberbach's conjecture, additional restrictions were imposed on functions of the class $\mathcal{S}$; these led to the introduction of several important subclasses of $\mathcal{S}$. The first subclass of $\mathcal{S}$ to be treated was that of *convex* functions introduced by Study [89] . These are functions which map Δ onto convex domains. The set of all convex functions in $\mathcal{S}$ is denoted by $\mathcal{S}^c$. A necessary and sufficient condition for a function $f(z) \in \mathcal{N}$ to belong to $\mathcal{S}^c$ is

$$(0.1.3) \qquad \mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > 0 \quad , \quad z \in \Delta \ .$$

Next to be considered was the class of *starlike* functions introduced by Alexander [3] . A domain D is said to be starlike with respect to a fixed point O in D if, for a given point P in D , the straight line segment OP also lies in D. We shall denote by S^* the class of functions in S which map Δ onto starlike domains with respect to the origin and by Σ^* the class of functions in Σ which map Δ onto domains whose complements are starlike with respect to the origin. A necessary and sufficient condition for a function $f(z) \in N$ to belong to S^* is

$$(0.1.4) \qquad \mathrm{Re}\{\frac{zf'(z)}{f(z)}\} > 0 \quad , z \in \Delta \quad .$$

A function g(z) with Laurent expansion (0.1.2) belongs to Σ^* if and only if

$$(0.1.5) \qquad \mathrm{Re}\{-\frac{zg'(z)}{g(z)}\} > 0 \ , \ z \in \Delta \quad .$$

The proofs for the conditions (0.1.3) , (0.1.4) and (0.1.5) may be found in Pommerenke [68, Chapter 2] .

In this thesis, we shall confine our attention to standard geometric properties such as distortion, covering and convexity of certain subclasses of starlike functions. Condition (0.1.4) suggests that these functions may be defined in terms of functions of positive real part in the unit disc. In fact, Janowski [33] introduced a general subclass of starlike functions in the following way.

Let B be the class of functions w(z) regular in Δ and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for $z \in \Delta$. We denote by $P(A, B)$, $-1 \leq B < A \leq 1$, the class of functions

$$(0.1.6) \qquad p(z) = 1 + p_1 z + p_2 z^2 + \ldots$$

defined by

$$(0.1.7) \qquad p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad , \quad z \in \Delta \quad ,$$

for some $w(z) \in \mathcal{B}$. The definition of this class $P(A, B)$ is a generalisation of the classical result (see Nehari [59, p. 169]) that any regular function $p(z) = 1+p_1z+p_2z^2+\ldots$ such that $\mathrm{Re}\{p(z)\} > 0$ in Δ can be written in the form

$$p(z) = \frac{1 + w(z)}{1 - w(z)} \quad , \quad w(z) \in \mathcal{B} \; . \tag{0.1.8}$$

The corresponding class $S^*(A, B)$ of starlike functions is now defined by

$$S^*(A, B) = \{f(z) \in N \quad ; \quad zf'(z)/f(z) \in P(A, B) \quad , \quad z \in \Delta\} \; .$$

The following special cases of $S^*(A, B)$ are of considerable interest:

$$S^*(1-2\alpha, -1) \equiv S^*_\alpha = \{f(z) \in N; \mathrm{Re}\{zf'(z)/f(z)\} > \alpha \; , \; 0 \le \alpha < 1 \; , \; z \in \Delta\} \; ,$$

$$S^*(1,1/M-1) \equiv S^*(M) = \{f(z) \in N; |zf'(z)/f(z)-M| < M \; , \; M > \tfrac{1}{2} \; , \; z \in \Delta\} \; ,$$

$$S^*(\alpha,0) \equiv S^*_{(\alpha)} = \{f(z) \in N; |zf'(z)/f(z)-1| < \alpha \; , \; 0 < \alpha \le 1 \; , \; z \in \Delta\} \; ,$$

$$S^*(\alpha,-\alpha) \equiv S^*[\alpha] = \{f(z) \in N; |zf'(z)/f(z)-1|/|zf'(z)/f(z)+1| < \alpha \; , \; 0 < \alpha \le 1 \; , \; z \in \Delta\} \; .$$

The class S^*_α of starlike functions of order α was introduced by Robertson [73] and has been extensively investigated. We note that $S^*_0 \equiv S^*$. Apart from $\alpha = 0$, the most interesting case is $\alpha = \frac{1}{2}$. Strohhäcker [88] and Marx [49] established that $S^c \subseteq S^*_{\frac{1}{2}}$ and the function $f(z) = z/(1+z)$ shows that the constant $\frac{1}{2}$ cannot be improved. The class $S^*(M)$ was first studied by Janowski [32]. Letting $M \to \infty$, $S^*(M)$ becomes S^* . The special case $M = 1$ was examined by Singh [84] . The class $S^*(1)$ was generalised to $S^*_{(\alpha)}$ by Wright [97], Bajpai [7] , Eenigenburg [19] and McCarty [52]. The class $S^*[\alpha]$ was introduced by Padmanabhan [62] . Here, the order of starlikeness was proposed in a way different from that defined by Robertson as we have seen above. However, putting

$\alpha = 1$, $S^*[\alpha]$ again coincides with S^* .

Tuan and Anh [93] generalised the classes S^*_α , $S^*(M)$ and $S^*_{(\alpha)}$ to

$S^*_{\beta,\gamma} = \{f(z) \in \mathbb{N}; |zf'(z)/f(z)-\beta| < \gamma \ , \ 0 < \gamma \le \beta \ , \ z \in \Delta\}$.

We note that $S^*_{\beta,\gamma}$ becomes S^*_α , $\alpha = \beta-\gamma$, when $\gamma \to \infty$. Now, put $\psi(z) = [zf'(z)/f(z)-\beta]/\gamma$; then $|\psi(z)| < 1$ in Δ and $\psi(0) = (1-\beta)/\gamma = B$. We next define $w(z) = [\psi(z)-B]/[1-B\psi(z)]$; then it is clear that $w(z) \in \mathcal{B}$ and, in terms of this $w(z)$, we have

$$\frac{zf'(z)}{f(z)} = \frac{1+(\gamma+\beta B)w(z)}{1+Bw(z)} \ .$$

Consequently, $S^*_{\beta,\gamma} \equiv S^*(A, B)$ with $A = \gamma+\beta B$, $B = (1-\beta)/\gamma$.

Recently, Mogra and Juneja [58] generalised $S^*_{(\alpha)}$ and, in particular, Padmanabhan's class $S^*[\alpha]$ to

$$S^*_{(\alpha,\beta)} = \{f(z) \in \mathbb{N}; |\frac{zf'(z)}{f(z)} -1|/|2\beta(\frac{zf'(z)}{f(z)} -\alpha) - (\frac{zf'(z)}{f(z)} -1)| < 1 \ , \\ 0 \le \alpha < 1 \ , \ 0 < \beta \le 1 \ , \ z \in \Delta\} \ .$$

However, with an argument as above, we may deduce that the characterising condition for $S^*_{(\alpha,\beta)}$ is

$$\frac{zf'(z)}{f(z)} = \frac{1+(1-2\alpha\beta)w(z)}{1+(1-2\beta)w(z)} \ , \quad w(z) \in \mathcal{B} \ .$$

Hence, once again, this is a special case of Janowski's class $S^*(A, B)$ with $A = 1-2\alpha\beta$, $B = 1-2\beta$. In view of these remarks, we see that a study of the class $S^*(A, B)$ leads to unified results on properties of various subclasses of starlike functions.

In this thesis, apart from $S^*(A, B)$, we shall also look at two of its subclasses, namely, the class $S^*_k(A, B)$ of functions in $S^*(A, B)$

with k-fold symmetric expansion:

$$S_k^*(A, B) = \{f(z)=z+a_{k+1}z^{k+1}+a_{2k+1}z^{2k+1}+\dots; zf'(z)/f(z) \in P_k(A, B), z \in \Delta\},$$

where

$$P_k(A, B) = \{p(z)=1+p_k z^k+p_{2k}z^{2k}+\dots \in P(A, B) \quad , \quad k = 1,2,3\dots\}$$

and the class $S_b^*(A, B)$ of functions in $S^*(A, B)$ with fixed second coefficient:

$$S_b^*(A, B) = \{f(z)=z+b(A-B)z^2+\dots; \ zf'(z)/f(z) \in P_b(A, B) \ , \ z \in \Delta\} \quad ,$$

where

$$P_b(A, B) = \{p(z) \in P(A, B) \ ; \ p'(0) = b(A-B) \ , \ 0 \le b \le 1\} \quad .$$

We shall also consider the subclass $S^*[a, b]$ defined by

$$S^*[a, b] = \{f(z) \in N \ ; \ zf'(z)/f(z) \in P[a, b] \ , \ z \in \Delta\} \quad ,$$

where

$$P[a, b] = \{p(z) \in P = P(1,-1) \ ; \ p(a) = b \ , \ 0 < a < 1 \ , \ b > 0\} \quad .$$

All the above-mentioned subclasses of S^* have their counterparts in $\sum^*$. However, since the analysis does not require any new consideration, we shall only treat the class

$$\sum^*(A, B) = \{f(z) = \frac{1}{z} + a_0 + a_1 z + \dots \ ; \ -zf'(z)/f(z) \in P(A, B) \ , \ z \in \Delta\}$$

and, in particular,

$$\sum^*(\alpha) = \{f(z) = \frac{1}{z} + a_0 + a_1 z + \dots; (1-\frac{\alpha}{2})\pi < \arg\{\frac{zf'(z)}{f(z)}\} < (1+\frac{\alpha}{2})\pi \ , \quad 0 < \alpha \le 1 \ , \ z \in \Delta\} \quad ,$$

the class of meromorphic strongly starlike functions of order α

introduced by Brannan, Clunie and Kirwan [12] . The class $\sum^*(A, B)$ with A, B subject to more restricted conditions that $-1 \le B < 0$, $B < A \le -B$ was considered by Karunakaran [35] . We shall be interested in the class $\sum^*(A, B)$ under the general conditions that $-1 \le B < A \le 1$.

Problems associated with subclasses of starlike functions may be transformed into those of minimising or maximising on $|z| = r < 1$ functionals of the form

$$\text{(0.1.9)} \qquad \operatorname{Re}\{F(p(z) , zp'(z))\} \quad ,$$

where F(u,v) is a given function regular in the v-plane and in the half-plane Re u > 0 and p(z) varies in some subclass of P . For example, we wish to find the radius of convexity of the class $S_k^*(A, B)$, that is the number

$$r_c = \sup\{r \; ; \operatorname{Re}\{1+ \frac{zf''(z)}{f'(z)}\} > 0 \; , \; |z| < r \; , f(z) \in S_k^*(A, B)\} \; .$$

The compactness of the class $S_k^*(A, B)$ implies that r_c is equal to the smallest root in (0,1] of the equation $\Omega(r) = 0$, where

$$\Omega(r) = \min\{\operatorname{Re}\{1+ \frac{zf''(z)}{f'(z)}\} \; ; \; |z| = r < 1 \; , \; f(z) \in S_k^*(A, B)\} \; .$$

From the definition of $S_k^*(A, B)$, we deduce that

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)} \quad , \quad p(z) \in P_k(A, B) \quad .$$

The problem is therefore resolved if we can find

$$\min_{p(z)\in P_k(A, B)} \; \min_{|z|=r<1} \; \operatorname{Re}\{p(z)+ \frac{zp'(z)}{p(z)}\} \quad .$$

As a further example, we wish to derive distortion bounds for the class $\sum^*(A, B)$. From the equation

$$\log(z^2f'(z)) = \log|z^2f'(z)| + i\arg(z^2f'(z))$$

and the definition of $\sum^*(A, B)$, we find

$$r\frac{\partial}{\partial r}\log|z^2f'(z)| = 1-\text{Re}\{p(z)-\frac{zp'(z)}{p(z)}\}\ , \quad p(z)\in P(A, B)\ .$$

Thus the problem is now reduced to finding

$$\min_{p(z)\in P(A, B)}\ \min_{|z|=r<1}\ \text{Re}\{p(z)-\frac{zp'(z)}{p(z)}\}\quad .$$

In this thesis we shall derive bounds on $|z| = r < 1$ for functionals of the form

(0.1.10) $\qquad \text{Re}\{\alpha p(z) + \beta zp'(z)/p(z)\}\ , \quad \alpha,\beta$ real ,

where $p(z)$ varies in each of the families $P(A, B)$, $P_k(A, B)$, $P_b(A, B)$ and $P[a, b]$.

Various methods have been developed to deal with extremal problems over P and its subclasses. The first and most significant result on problems of the form (0.1.9) for the class P was that given by Robertson [75] . Based upon the variational formula for functions $p(z) \in P$ that

$$p^*(z) = p(z) - \rho^2(1-|z_0|^2)\lambda(z,z_0) + o(\rho^2)\quad ,$$

where

$$\lambda(z,z_0) = [\frac{z_0p'(z)}{p(z_0)} -1]\frac{ze^{i\theta}}{z_0(z_0-z)} +[\frac{z_0p(z)}{p(z_0)} -z]\ \frac{ze^{i\theta}}{z_0(z_0-z)^2}$$

$$+ \frac{p'(z)}{p(z_0)}\cdot\frac{z^2e^{-i\theta}}{1-\bar{z}_0z} +[\frac{p(z)}{p(z_0)} +1]\frac{ze^{-i\theta}}{(1-\bar{z}_0z)^2}\quad ,$$

$|z| < 1$, θ real and arbitrary, ρ real and small (see Robertson [74]) , Robertson proved

0.1.1 Theorem [75] . *Let* $F(u,v)$ *be regular in the* v*-plane and in the half-plane* $\operatorname{Re} u > 0$*; then for every* r , $0 < r < 1$, *the value of*

$$\min_{p(z)\in P} \min_{|z|=r} \operatorname{Re}\{F(p(z) , zp'(z))\}$$

occurs only for a function of the form

$$(0.1.11) \qquad p(z) = \frac{1+\alpha}{2} \cdot \frac{1+ze^{i\theta}}{1-ze^{i\theta}} + \frac{1-\alpha}{2} \cdot \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} ,$$

where $-1 \le \alpha \le 1$, $0 \le \theta \le 2\pi$.

Thus, to solve an extremal problem such as

$$(0.1.12) \qquad \min_{|z|=r<1} \operatorname{Re}\{p(z)+ \frac{zp'(z)}{p(z)}\}$$

over P , we only have to substitute into (0.1.12) the function $p(z)$ given by (0.1.11) and to find the minimum of the resulting function of three real variables. However, this is precisely where the difficulties lie (see Robertson [75, Theorem 3] and Libera [38, Theorem 1]) . Zmorovič [100] devised an ingenious, but quite simple, technique to overcome these difficulties. This is described in the following

0.1.2 Theorem [100] . *Let* $p(z)$ *be as given by* (0.1.11) *; then* $zp'(z)$ *can be written in the form*

$$(0.1.13) \qquad zp'(z) = \tfrac{1}{2}(p(z)^2-1) + \tfrac{1}{2}(\rho^2-\rho_0^2)e^{2i\psi} ,$$

where $(1+\varepsilon_k z)/(1-\varepsilon_k z) = a+\rho e^{i\psi_k}$, $k = 1,2$, $\varepsilon_1 = e^{i\theta}$, $\varepsilon_2 = e^{-i\theta}$,

$p(z) = a+\rho_0 e^{i\psi_0}$, $0 \le \rho_0 \le \rho$, $a = (1+r^2)/(1-r^2)$, $\rho = 2r/(1-r^2)$,
$e^{i\psi} = ie^{i(\psi_1+\psi_2)/2}$.

For a fixed value of p(z) in the disc $|p(z)-a| \le \rho$, the angle 2ψ in (0.1.13) can take any value in the interval $[0,2\pi]$ as shown by Zmorovič [100] . Hence if we put

$$F(u,v) = M(u) + N(u).v \quad ,$$

where M(u) , N(u) are regular in the half-plane Re $u > 0$, $u = p(z)$, $v = zp'(z)$, p(z) being as given by (0.1.11), then it follows from (0.1.13) that

$$(0.1.14) \quad \min \operatorname{Re}\{F(u,v)\} = \operatorname{Re}\{M(u)+\tfrac{1}{2}(u^2-1)N(u)\} - \tfrac{1}{2}|N(u)|(\rho^2-\rho_0^2) \quad .$$

This minimum is reached when

$$(0.1.15) \qquad \exp[i(2\psi+ \arg N(u)] = -1 \ .$$

In view of Robertson's Theorem 0.1.1 and equation (0.1.14) , problem (0.1.12) is reduced to finding the minimum of a function of u in the disc $|u-a| \le \rho$, which is a significant simplification. Zmorovič employed this technique to solve completely the problems of determining the radii of convexity of S^*_α and $\sum^*_\alpha \equiv \sum^*(1-2\alpha,-1)$, $0 \le \alpha < 1$.

Returning to our class of interest $P(A, B)$, it can be easily shown that if $q(z) \in P(A, B)$, then

$$(0.1.16) \qquad q(z) = \frac{(1+A)p(z) + 1-A}{(1+B)p(z) + 1-B} \quad ,$$

for some $p(z) \in P$ and conversely. We denote by $P(A, B)_2$ the subclass of $P(A, B)$ containing all functions of the form (0.1.16) where p(z) is given by (0.1.11) . Then Robertson's result and the representation (0.1.16) imply that the functions which minimise the functionals Re{F(p(z),zp'(z))} over $P(A, B)$ must belong to $P(A, B)_2$. Hence extremal problems for these functionals can be replaced by analogous problems over $P(A, B)_2$. Equipped with this fact, Janowski extended Zmorovič's technique (see [33,Lemma 4]) and solved the problems min Re$\{p(z)+zp'(z)/p(z)\}$ and min Re$\{zp'(z)/p(z)\}$ over $P(A, B)$. The analysis is, however, lengthy and extremely complicated. In the face of these complications, no result is known so far concerning extremal problems over $P_k(A, B)$ or $P_b(A, B)$ through the use of Robertson-Zmorovič's method.

For the classes of functions which have an integral representation, the solution of extremal problems is relatively simpler. We want to mention next Golusin's method.

The Herglotz integral representation formula for functions $p(z)$ of the class P is given by (see Pommerenke [68,p.40])

$$p(z) = \int_{-\pi}^{\pi} \frac{1+e^{-it}z}{1-e^{-it}z}\, d\mu(t) \quad ,$$

where $\mu(t) \in I$, the class of non-decreasing functions on $[-\pi,\pi]$ with total variation $\int_{-\pi}^{\pi} d\mu(t) = 1$. This formula provides integral representations for various subclasses of S and Σ . For instance, a function $f(z)$ in S^* can be represented by (see Pommerenke [68,p.43])

$$f(z) = z \exp\int_{-\pi}^{\pi} \log(1-e^{-it}z)^{-2} d\mu(t) \quad , \quad \mu(t) \in I \quad .$$

These representations are in general of the form $\int_{-\pi}^{\pi} g(z,t)d\mu(t)$, where $g(z,t)$ is a given kernel depending on the subclass and $\mu(t)$ varies in I . The solving of extremal problems for these classes is therefore equivalent to finding the functions $\mu(t)$ which correspond to the extremal functions $f(z)$. In this direction, the following two variational formulae proposed by Golusin |26| have proved to be effective.

Let E_g denote the class of regular functions having the representation $\int_{-\pi}^{\pi} g(z,t)d\mu(t)$, where $g(z,t)$ is a fixed function regular in Δ for each t in $[-\pi,\pi]$ and $\mu(t) \in I$.

0.1.3 <u>Theorem</u> [24] . *Let* $f(z) \in E_g$, t_1, t_2 *be given with* $-\pi \le t_1 < t_2 < \pi$ *and* λ *be any number in* $[-1,1]$. *Then there exists a real constant* c *independent of* λ *and* t *such that the functions*

$$(0.1.17) \qquad f_*(z) = f(z) + \lambda\int_{t_1}^{t_2} \frac{\partial g(z,t)}{\partial t} |\mu(t)-c|dt$$

are also in E_g .

0.1.4 <u>Theorem</u> [24] . *Let* $f(z) \in E_g$ *and* t_1, t_2 *with* $-\pi \le t_1 < t_2 < \pi$ *be two jump points for the function* $\mu(t)$. *Then there exists a number* $\eta > 0$ *such that for all* λ *in* $(-\eta,\eta)$, *the functions*

$$(0.1.18) \qquad f_{**}(z) = f(z) + \lambda[g(z,t_1) - g(z,t_2)]$$

are also in E_g .

In particular, these two simple variational formulae may be used to establish Robertson's Theorem 0.1.1 stated above (see Pfaltzgraff and Pinchuk [64, Theorem 7.3]). Thus this method supplies an alternative approach to solving extremal problems for the class P and certain subclasses of S , for example, S_α^* and S_α^c (see Pinchuk [65]). However, in dealing with subclasses of $\sum$ such as $\sum^*$ and $\sum^c$, the Golusin's variational formulae are no longer applicable. The integral representations for functions of these subclasses also have the form $\int_{-\pi}^{\pi} g(z,t)d\mu(t)$, but $\mu(t)$ now must satisfy certain constraints of the form $\int_{-\pi}^{\pi} e^{-it} d\mu(t) = \gamma$, γ being a fixed number. These constraints are not invariant by Golusin's technique. To remove this difficulty, Pfaltzgraff and Pinchuk developed a new variational method which extends Golusin's and preserves the constraints. For further details, the reader is referred to the original paper of Pfaltzgraff and Pinchuk [64] .

In the same paper, these authors considered extremal problems for two subclasses of P, namely, $P_b \equiv P_b(1,-1)$ and $P[a, b]$. However, they chose to approach these problems by deriving representation formulae for P_b and $P[a, b]$ in terms of functions of P and applying directly the original Golusin's variations (0.1.17) and (0.1.18). This method yields better results than those obtained by using their new variational formulae . They proved

0.1.5 Theorem [64]. *Let* $F(u,v)$ *be regular in the* v*-plane and the half-plane* $\operatorname{Re} u > 0$. *Then the functional* $\operatorname{Re}\{F(p(z),zp'(z))\}$ *attains its maximum or minimum on* $|z| = r$, $0 < r < 1$, *over the class* P_b *or* $P[a, b]$ *only for functions of the form*

$$p(z) = \sum_{k=1}^{n+1} \beta_k (e^{it_k}+z)/(e^{it_k}-z) \quad , \tag{0.1.19}$$

where $n \le 2$, $\beta_k \ge 0$, $\sum_{k=1}^{n+1} \beta_k = 1$ *and*

(i) $\sum_{k=1}^{n+1} \beta_k e^{-it_k} = b$ *for* $p(z) \in P_b$,

(ii) $\sum_{k=1}^{n+1} \beta_k (e^{it_k}+a)/(e^{it_k}-a) = b$ *for* $p(z) \in P[a, b]$.

We observe that substituting the function (0.1.19) into (0.1.12) would reduce (0.1.12) to a problem of minimising a function of as many as 2n+1 , $n \leq 2$, real variables, which in most cases is a difficult task. Hence, once again, as far as these subclasses of P are concerned, Golusin-Pfaltzgraff-Pinchuk's variational techniques would hardly be of any use.

To tackle extremal problems for $P[a, b]$ and the more general class $P_b(A, B)$, we shall take a different course, the classical approach, which is simpler and independent of variational techniques. The basic idea of the method is representing functions p(z) of the class under consideration in terms of functions of B and making use of Dieudonné's inequality

$$(0.1.20) \qquad |zw'(z) - w(z)| \leq \frac{|z|^2 - |w(z)|^2}{1 - |z|^2} \ , \quad w(z) \in B \ ,$$

or its equivalent (see Lemma 1.2.1 and (2.2.1)). This idea is due to Singh and Goel [85] who successfully solved the extremal problems

$$\min_{|z|=r<1} \operatorname{Re}\{p(z) \pm zp'(z)/p(z)\}$$

over $P_\alpha \equiv P(1-2\alpha,-1)$, $0 \leq \alpha < 1$.

In order to describe the method in some detail, let us again choose the extremal problem (0.1.12) with p(z) varying in $P(A, B)$ as an example. This method presents a typical approach to extremal problems of the form (0.1.10) for all the classes $P(A, B)$, $P_k(A, B)$, $P_b(A, B)$ and $P[a, b]$. Our starting point is the representation formula (0.1. 7) from which we deduce that

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{1+Aw(z)}{1+Bw(z)} + (A-B)\frac{zw'(z)}{[1+Aw(z)][1+Bw(z)]} \ .$$

Now, in view of (0.1.20) we find

$$(0.1.21) \qquad \mathrm{Re}\{p(z)+\frac{zp'(z)}{p(z)}\} \geq \mathrm{Re}\{\frac{1+Aw(z)}{1+Bw(z)} + \frac{(A-B)w(z)}{[1+Aw(z)][1+Bw(z)]}\}$$
$$- \frac{(A-B)(|z|^2 - |w(z)|^2)}{(1-|z|^2)|1+Aw(z)||1+Bw(z)|} .$$

From (0.1. 7) we also have, for $w(z) \in \mathcal{B}$,

$$w(z) = \frac{p(z)-1}{A-Bp(z)} \quad , \quad p(z) \in P(A, B) \quad , \quad z \in \Delta \quad .$$

Hence, in terms of $p(z)$, inequality (0.1.21) becomes

$$(0.1.22) \quad \mathrm{Re}\{p(z)+\frac{zp'(z)}{p(z)}\} \geq \frac{A+B}{A-B} + \frac{1}{A-B}\mathrm{Re}\{(A-2B)p(z) - \frac{A}{p(z)}\}$$
$$- \frac{|z|^2|A-Bp(z)|^2 - |p(z)-1|^2}{(A-B)(1-|z|^2)|p(z)|} .$$

Thus, the solution to problem (0.1.12) may be obtained on minimising the right-hand side of (0.1.22) when $p(z)$ takes its values in some disc $|p(z)-a| \leq d$ which we shall determine later. In most cases, the minimum is reached on the diameter of this disc. However, this is the most difficult part of the analysis. Once this difficulty is overcome, we are left with only one further parameter to determine.

We now sketch the plan of the thesis.

In Chapter 1, the classical method which we have just described is employed to derive the lower bound on $|z| = r < 1$ for the functional

$$(0.1.23) \qquad \mathrm{Re}\{\alpha p(z) + \beta\frac{zp'(z)}{p(z)}\} \quad , \quad \alpha \geq 0 \ , \quad \beta \geq 0 \quad ,$$

over $P(A, B)$. The work of Janowski [33] on the class $P(A, B)$

corresponds to the cases $\alpha = 1$, $\beta = 1$ and $\alpha = 0$, $\beta = 1$. Thus we not only give simpler proofs to Janowski's results, but also combine and generalise them to some extent. For some applications of (0.1.23) we determine the radius of starlikeness of functions $f(z) \in N$ for which $f(z)/g(z) \in P(A, B)$, $g(z) \in S_\alpha^*$ and the radius of starlikeness of functions $k(z) = \lambda f(z) + (1-\lambda)z$, $0 \leq \lambda \leq 1$, where $f(z) \in N$ with $\mathrm{Re}\{f(z)/z\} > \frac{1}{2}$ in Δ . The solution for the former problem unifies and generalises several known results, and in particular, answers some questions left open by Shaffer in [82]. The latter problem is due to Trimble [91] .

Chapter 2 deals with the functional (0.1.23) over the class $P_k(A, B)$. The result is useful in studying properties such as distortion , covering and convexity of the corresponding class $S_k^*(A, B)$ of k-fold symmetric starlike functions. As another application of (0.1.23) with $p(z) \in P_k(A, B)$, we derive the radius of starlikeness of functions $\lambda f(z) + (1-\lambda)zf'(z)$, $-\infty < \lambda < 1$, where $f(z) \in S_k^*(A, B)$. This problem was first investigated by Livingston [42] for the case $\lambda = \frac{1}{2}$, $f(z) \in S^*$.

In Chapter 3, we shall be concerned with the functional (0.1.23) when $p(z)$ varies in $P_b(A, B)$. The influence of the second coefficient in the series expansion of a function upon the behaviour of that function is examined for two classes of regular functions generated from $P_b(A, B)$, namely, $S^*(A, B)$ and

$$P_b'(A, B) = \{f(z) = z + \frac{b}{2}(A-B)z^2 + \ldots;\ \ f'(z) \in P_b(A, B)\ ,\ \ z \in \Delta\} \ .$$

The results obtained generalise those due to Finkelstein [20] , Tepper [90] and McCarty [51] . This chapter also establishes the radius of convexity of the subclass of close-to-starlike functions

$$R_b = \{f(z) = z-2bz^2+\dots \; ; \; 0 \le b \le 1 \; , \; \mathrm{Re}\{f(z)/z\} > 0 \; , \; z \in \Delta\}$$

involving the pre-assigned second coefficient in the series expansion of functions in the class. This radius refines that found by Reade, Ogawa and Sakaguchi [72] for the class

$$R = \{f(z) \in N \; ; \; \mathrm{Re}\{f(z)/z\} > 0 \quad , \quad z \in \Delta\} \quad .$$

The functional (0.1.23) with $p(z)$ varying in $P[a, b]$ is studied in Chapter 4 . We shall give the sharp lower bound for (0.1.23) when $b > 1$. We shall also obtain the lower bound on $|z| = r < 1$ for the functional $\mathrm{Re}\{p(z) + zp'(z)\}$ over $P[a, b]$ when $b > 1$. These results will then be used to derive certain distortion properties for two subclasses of regular functions associated with $P[a, b]$, namely, $S^*[a, b]$ and

$$R[a, b] = \{f(z) \in N \; ; \; f(z)/z \in P[a, b] \; , \quad z \in \Delta\} \; .$$

In Chapter 5 we continue our investigation of the functional (0.1.23) over $P(A, B)$ by looking at the case $\alpha \le 1$, $\beta = -1$. The aim of this consideration is to investigate the class $\sum^*(A, B)$. However, the distortion theorem for $\sum^*(A, B)$ is far from being complete. Instead, distortion bounds for certain special cases of $\sum^*(A, B)$ are determined . In this chapter, we also find sharp bounds on $|z| = r < 1$ for the functional

$$\mathrm{Re}\{p(z)^\alpha - \alpha\frac{zp'(z)}{p(z)}\} \quad , \quad 0 < \alpha \le 1 \; ,$$

over P . These bounds are necessary for the examination of distortion, covering and convexity of the class $\sum^*(\alpha)$ of meromorphic strongly starlike functions of order α .

Chapter 6 treats the functional (0.1.23) with $\alpha = 0$, $\beta = 1$ over the class $P(A, B)$ with $-1 < A < \infty$, $-1 \leq B < 1$, $B < A$. The case in which we may have $A \geq 1$ arises from the investigation of the starlikeness of functions $f(z) \in N$ and satisfying in Δ the inequality

$$\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)g(z)} - \gamma\right| < \gamma \ , \ \gamma \geq 1 \ , \ 0 \leq \lambda < 1 \ ,$$

where $g(z)$ belongs to S_α^* , for example. For $A > 1$, the function $zp'(z)/p(z)$ is no longer regular in the entire unit disc. Hence we shall restrict λ in some smaller range so that the regularity of $zp'(z)/p(z)$ is restored. We shall consider $g(z)$ varying in each of the families R , $R_{\frac{1}{2}}$, S_α^* and S , where

$$R_{\frac{1}{2}} = \{f(z) \in N \ ; \ \mathrm{Re}\{f(z)/z\} > \tfrac{1}{2} \ , \ \ z \in \Delta\} \ .$$

Our results sharpen and generalise those given by Shah [83] .

In Chapter 7 we complete our study on the functional (0.1.23) over $P(A, B)$ by considering the case $\alpha = 1$, $\beta < 0$. This together with the bound for (0.1.23) established in Chapter 1 are employed to examine the β-convexity of the class $S^*(A, B)$. The concept of β-convexity, introduced by Mocanu [56] , generalises those of convexity and starlikeness. Thus, a function $f(z) \in N$ with $f(z)f'(z)/z \neq 0$ in Δ is said to be β-convex, β real, if it satisfies the inequality

$$\mathrm{Re}\{(1-\beta)\frac{zf'(z)}{f(z)} + \beta(1+ \frac{zf''(z)}{f'(z)})\} > 0 \ , \ \ z \in \Delta \ .$$

We shall give complete results on the β-convexity of the classes S_α^* , $0 \leq \alpha \leq \frac{1}{2}$, and $S^*[\alpha]$. Corresponding results for the meromorphic cases, that is, for the classes $\sum_\alpha^*$, $0 \leq \alpha \leq \frac{1}{2}$, and $\sum^*[\alpha]$ are also obtained.

0.2 Some definitions and known results

0.2.1 In this thesis, the terms normal and compact will be used in the following sense as defined by Nehari [59, pp. 140-141] .

A family $\mathcal{F}$ of regular functions in a domain D is said to be *normal* if any sequence $\{f_n\}$ of functions in $\mathcal{F}$ contains a subsequence $\{f_{n_k}\}$ which converges uniformly in any closed subdomain of D.

A normal family $\mathcal{F}$ of functions is said to be *compact* if the limits of all convergent sequences of functions in $\mathcal{F}$ also belong to $\mathcal{F}$.

It is well-known that the classes $\mathcal{S}$ and $\mathcal{P}$ are normal and compact (see Nehari [59, pp. 217 and 143]).

0.2.2 For a function $f(z) \in \mathcal{S}$, we have (see Nehari [59, pp.214-217])

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2} \quad , \tag{0.2.1}$$

$$\frac{1-r}{(1+r)^2} \le |f'(z)| \le \frac{1+r}{(1-r)^2} \quad , \quad |z| = r < 1 \quad ; \tag{0.2.2}$$

$$f(\Delta) \supseteq \{w \; ; \; |w| < \tfrac{1}{4}\} \quad . \tag{0.2.3}$$

Inequalities (0.2.1) and (0.2.2) are collectively known as the distortion theorem for $\mathcal{S}$, while (0.2.3) is the covering theorem for $\mathcal{S}$.

0.2.3 When we say that a function f(z) is starlike without referring to its star centre point, we shall mean that f(z) is starlike with respect to the origin.

By the radius of starlikeness of a subclass $\mathcal{F}$ of regular functions, we mean the least upper bound of the radii of the discs $|z| < r$ in which the functions of $\mathcal{F}$ are starlike. Since a function $f(z) \in \mathcal{F}$ is starlike in $|z| < r$ if and only if $\mathrm{Re}\{zf'(z)/f(z)\} > 0$ in $|z| < r$, the radius of starlikeness r_* of $\mathcal{F}$ is therefore equal to

$$r_* = \sup\{r \; ; \; \mathrm{Re}\{zf'(z)/f(z)\} > 0 \;, \; |z| < r \;, \; f(z) \in \mathcal{F}\} \;.$$

If $\mathcal{F}$ is compact, then r_* is equal to the smallest root in $(0,1]$ of the equation $\Omega(r) = 0$, where

$$\Omega(r) = \min_{f(z)\in\mathcal{F}} \; \min_{|z|=r} \; \mathrm{Re}\{zf'(z)/f(z)\} \;. \tag{0.2.4}$$

Analogously, the radius of convexity r_c of $\mathcal{F}$ is defined to be the least upper bound of the radii of the discs $|z| < r$ in which the functions of $\mathcal{F}$ are convex. The characterising condition (0.1.3) of S^c is now taken into account to give

$$r_c = \sup\{r \; ; \; \mathrm{Re}\{1+zf''(z)/f'(z)\} > 0 \;, \; |z| < r \;, \; f(z) \in \mathcal{F}\} \;.$$

If $\mathcal{F}$ is compact, then r_c is equal to the smallest root in $(0,1]$ of the equation $\Psi(r) = 0$, where

$$\Psi(r) = \min_{f(z)\in\mathcal{F}} \; \min_{|z|=r} \; \mathrm{Re}\{1+zf''(z)/f'(z)\} \;. \tag{0.2.5}$$

In a similar way, we define the radius of convexity of a subclass $\mathcal{G}$ of Σ to be the number

$$\rho = \sup\{r \; ; \; \mathrm{Re}\{-(1+zf''(z)/f'(z))\} > 0 \;, \; |z| < r \;, \; f(z) \in \mathcal{G}\} \;.$$

0.2.4 A function $f(z) \in \mathcal{N}$ is said to be close-to-convex in Δ if there exists a function $g(z) \in S^c$ such that

$$\mathrm{Re}\{e^{i\theta} \frac{f'(z)}{g'(z)}\} > 0 \;, \; 0 \le \theta < 2\pi \;, \; z \in \Delta \;. \tag{0.2.6}$$

From the well-known relationship (see Pommerenke [68, p. 46]) that

$f(z) \in S^c$ if and only if $zf'(z) \in S^*$, condition (0.2.6) can be replaced by

$$(0.2.7) \qquad \mathrm{Re}\{e^{i\theta} \frac{zf'(z)}{g(z)}\} > 0 \ , \ 0 \le \theta < 2\pi \ , \ z \in \Delta \ ,$$

for some $g(z) \in S^*$. This class of close-to-convex functions, denoted by S^{cc}, was introduced by Kaplan [34]. It is clear from (0.2.7) that every starlike univalent function is close-to-convex. Kaplan [34] further showed that every close-to-convex function is univalent. Hence we have the inclusion relations

$$(0.2.8) \qquad S^c \subseteq S^* \subseteq S^{cc} \subseteq S \ .$$

The Bieberbach's conjecture for S^{cc} was proved by Reade [71]. Thus, if $f(z) = z+a_2z^2+a_3z^3+\ldots \in S^{cc}$, then $|a_n| \le n$, $n = 2,3,4,\ldots$ In the same paper, [71], Reade introduced the class of close-to-starlike functions, which we denote by S^{c*}. A function $f(z) \in \mathbb{N}$ is said to be close-to-starlike in Δ if there exists a function $g(z) \in S^*$ for which

$$(0.2.9) \qquad \mathrm{Re}\{e^{i\theta} \frac{f(z)}{g(z)}\} > 0 \ , \ 0 \le \theta < 2\pi \ , \ z \in \Delta \ .$$

The relationship between S^{c*} and S^{cc} is the same as that between S^* and S^c, that is, $f(z) \in S^{cc}$ if and only if $zf'(z) \in S^{c*}$. However, unlike the close-to-convex functions, the functions of the class S^{c*} are not necessarily univalent. An easy example is the function $f(z)=z(1+z)/(1-z)$ which is in S^{c*} but not univalent in any disc of radius $r > 2-\sqrt{3}$ since $f'(2-\sqrt{3}) = 0$.

0.2.5 Let $f(z)$ be regular in Δ and $g(z)$ be regular and univalent in Δ. We say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$, in Δ if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$.

The univalence of $g(z)$ and the condition $f(0) = g(0)$ imply that the function $w(z)$ defined by $w(z) = g^{-1}(f(z))$ is in $\mathcal{B}$ and $f(z) = g(w(z))$. By Schwarz's lemma, we have $|w(z)| \leq |z|$ in Δ . Hence

$$\{f(z) \ ; \ |z| < r\} \subset \{g(z) \ ; \ |z| < r\} \ , \quad 0 < r < 1 \ .$$

This proves

The Subordination Principle. *Let* $f(z)$, $g(z)$ *be regular in* Δ . *If* $g(z)$ *is univalent in* Δ , *then the conditions* $f(0) = g(0)$ *and* $f(\Delta) \subseteq g(\Delta)$ *imply* $f(\Delta_r) \subseteq g(\Delta_r)$, *where* $\Delta_r = \{z \ ; \ |z| < r \ , \ 0 < r < 1\}$.

0.2.6 For a function $f(z) \in S^*(A, B)$, it follows from the relation $zf'(z)/f(z) = p(z)$ for $p(z) \in P(A, B)$ that

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{1}{z}[p(z)-1] \ .$$

Hence, on integrating both sides, we get

$$\log\frac{f(z)}{z} = \int_0^z [p(\xi)-1]\frac{d\xi}{\xi} \ ,$$

that is,

$$f(z) = z \exp\int_0^z [p(\xi)-1]\frac{d\xi}{\xi} \ . \tag{0.2.10}$$

This is the structural formula for $S^*(A, B)$. The structural formulae for the classes $S_k^*(A, B)$, $S_b^*(A, B)$, $S^*[a, b]$ are also given by (0.2.10) with $p(z)$ now belongs to $P_k(A, B)$, $P_b(A, B)$, $P[a, b]$ respectively .

0.3 List of classes of functions

We list in this section the classes of functions frequently encountered in the thesis. Here and throughout the thesis, the unit disc $\{z \,;\, |z| < 1\}$ is denoted by Δ .

$\mathcal{N}$, the class of regular functions $f(z)$ in Δ with series expansion
$f(z) = z+a_2z^2+a_3z^3+\ldots$

$\mathcal{B}$, the class of functions $w(z)$ regular in Δ and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for $z \in \Delta$.

$\mathcal{P} = \{p(z) = 1+p_1z+\ldots \,;\, \mathrm{Re}\{p(z)\} > 0 \,,\; z \in \Delta\}$.

$\mathcal{P}_\alpha = \{p(z) = 1+p_1z+\ldots \,;\, \mathrm{Re}\{p(z)\} > \alpha \,,\; 0 \le \alpha < 1 \,,\; z \in \Delta\}$.

$\mathcal{P}(A, B) = \{p(z) = 1+p_1z+\ldots \,;\, p(z) = \frac{1+Aw(z)}{1+Bw(z)} \,,\; -1 \le B < A \le 1 \,,\; w(z) \in \mathcal{B} \,,\; z \in \Delta\}$.

$\mathcal{P}' = \{f(z) \in \mathcal{N} \,;\, \mathrm{Re}\{f'(z)\} > 0 \,,\; z \in \Delta\}$.

$\mathcal{R} = \{f(z) \in \mathcal{N} \,;\, \mathrm{Re}\{f(z)/z\} > 0 \,,\; z \in \Delta\}$.

$\mathcal{R}_\alpha = \{f(z) \in \mathcal{N} \,;\, \mathrm{Re}\{f(z)/z\} > \alpha \,,\; 0 \le \alpha < 1 \,,\; z \in \Delta\}$.

$\mathcal{S}$, the class of functions $f(z) \in \mathcal{N}$ which are univalent in Δ .

$\mathcal{S}^c$, the class of functions $f(z) \in \mathcal{S}$ which are convex in Δ .

$\mathcal{S}^c_\alpha = \{f(z) \in \mathcal{N} \,;\, \mathrm{Re}\{1+zf''(z)/f'(z)\} > \alpha \,,\; 0 \le \alpha < 1 \,,\; z \in \Delta\}$.

$\mathcal{S}^*$, the class of functions $f(z) \in \mathcal{S}$ which are starlike in Δ .

$\mathcal{S}^*_\alpha = \{f(z) \in \mathcal{N} \,;\, \mathrm{Re}\{zf'(z)/f(z)\} > \alpha \,,\; 0 \le \alpha < 1 \,,\; z \in \Delta\}$.

$\mathcal{S}^*(M) = \{f(z) \in \mathcal{N} \,;\, |zf'(z)/f(z)-M| < M \,,\; M > \tfrac{1}{2} \,,\; z \in \Delta\}$.

$\mathcal{S}^*_{(\alpha)} = \{f(z) \in \mathcal{N} \,;\, |zf'(z)/f(z)-1| < \alpha \,,\; 0 < \alpha \le 1 \,,\; z \in \Delta\}$.

$\mathcal{S}^*[\alpha] = \{f(z) \in \mathcal{N} \,;\, |zf'(z)/f(z)-1|/|zf'(z)/f(z)+1| < \alpha \,,\; 0 < \alpha \le 1 \,,\; z \in \Delta\}$.

$S^*(\alpha) = \{f(z) \in \mathbb{N} \; ; \; |\arg\{zf'(z)/f(z)\}| < \alpha\pi/2 \; , \; 0 < \alpha \leq 1 \; , \; z \in \Delta\}$.

$S^*(A, B) = \{f(z) \in \mathbb{N} \; ; \; zf'(z)/f(z) \in P(A, B) \; , \; z \in \Delta\}$.

$S^{c*} = \{f(z) \in \mathbb{N} \; ; \; \mathrm{Re}\,\{e^{i\theta}f(z)/g(z)\} > 0 \; , \; g(z) \in S^* \; , \; 0 \leq \theta < 2\pi \; , \; z \in \Delta\}$.

$S^{cc} = \{f(z) \in \mathbb{N} \; ; \; \mathrm{Re}\{e^{i\theta}f'(z)/g'(z)\} > 0 \; , \; g(z) \in S^c \; , \; 0 \leq \theta < 2\pi \; , \; z \in \Delta\}$.

Σ , the class of functions $g(z)$ which are meromorphic and univalent in Δ with Laurent expansion $g(z) = 1/z+b_0+b_1z+\ldots$

Σ^c , the class of functions $g(z) \in \Sigma$ which map Δ onto domains whose complements are convex.

$\Sigma^c_\alpha = \{g(z) = 1/z+b_0+b_1z+\ldots \; ; \; \mathrm{Re}\{-(1+\frac{zg''(z)}{g'(z)})\} > \alpha \; , \; 0 \leq \alpha < 1 \; , \; z \in \Delta\}$.

Σ^* , the class of functions $g(z) \in \Sigma$ which map Δ onto domains whose complements are starlike with respect to the origin.

$\Sigma^*_\alpha = \{g(z) = 1/z+b_0+b_1z+\ldots \; ; \; \mathrm{Re}\{-(zg'(z)/g(z))\} > \alpha \; , \; 0 \leq \alpha < 1 \; , \; z \in \Delta\}$.

$\Sigma^*[\alpha] = \{g(z) = 1/z+b_0+b_1z+\ldots \; ; \; |(\frac{zg'(z)}{g(z)}+1)/(\frac{zg'(z)}{g(z)}-1)| < \alpha \; , \; 0 < \alpha \leq 1 \; , \; z \in \Delta\}$.

$\Sigma^*(\alpha) = \{g(z) = 1/z+b_0+b_1z+\ldots; \; (1-\frac{\alpha}{2})\pi < \arg\{\frac{zg'(z)}{g(z)}\} < (1+\frac{\alpha}{2})\pi \; , \; 0 < \alpha \leq 1 \; , \; z \in \Delta\}$.

$\Sigma^*(A, B) = \{g(z) = 1/z+b_0+b_1z+\ldots \; ; \; -zg'(z)/g(z) \in P(A, B) \; , \; z \in \Delta\}$.

CHAPTER 1

REGULAR FUNCTIONS OF POSITIVE REAL PART

1.1 Introduction

This chapter is concerned with extremal problems for the class $P(A, B)$ and selected applications.

By means of a variational formula for functions in P, Robertson [75] proved that if $F(u, v)$ is regular in the v- plane and in the half-plane $\operatorname{Re} u > 0$, then for every r, $0 < r < 1$, the value of

$$\min_{p(z)\in P} \min_{|z|=r} \operatorname{Re}\{F(p(z), zp'(z))\}$$

occurs only for a function of the form

$$p(z) = \frac{1+\alpha}{2} \cdot \frac{1+ze^{i\theta}}{1-ze^{i\theta}} + \frac{1-\alpha}{2} \cdot \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}, \tag{1.1.1}$$

where $-1 \le \alpha \le 1$, $0 \le \theta \le 2\pi$.

On the other hand, since every function $q(z)$ in $P(A, B)$ can be represented by

$$q(z) = \frac{(1+A)p(z) + 1 - A}{(1+B)p(z) + 1 - B}, \tag{1.1.2}$$

where $p(z) \in P$, it follows from Robertson's result that the functions which minimise the functionals $\operatorname{Re}\{F(p(z), zp'(z))\}$ over $P(A, B)$ must be of the form (1.1.2) with $p(z)$ as given by (1.1.1).

Making use of this observation, Janowski [33] found the upper and lower bounds for the functionals $\operatorname{Re}\{p(z) + zp'(z)/p(z)\}$ and

$\text{Re}\{zp'(z)/p(z)\}$ on $|z| = r < 1$, where $p(z)$ varies over $P(A, B)$. However, the analysis is lengthy and rather involved.

In this chapter, we give a short and simple solution for the more general problem

$$(1.1.3) \qquad \min_{|z|=r<1} \text{Re}\{\alpha\, p(z) + \beta\, zp'(z)/p(z)\}\ , \ \alpha \geq 0,\ \beta \geq 0\ ,$$

over $P(A, B)$. Janowski's results correspond to the cases $\alpha = 1$, $\beta = 1$ and $\alpha = 0$, $\beta = 1$ respectively. Our approach is classical and independent of variational techniques.

The problem (1.1.3) has many applications in the theory of subclasses of univalent functions, in particular, those which are characterised by some relationship with functions of positive real part such as S^*, S^c. To illustrate a few of these applications, we investigate the following two problems:

(i) the radius of starlikeness of functions $f(z) \in N$ which satisfy $f(z)/g(z) = p(z)$ in Δ, where $g(z)$ belongs to some subclass of S or N and $p(z) \in P(A, B)$;

(ii) the radius of starlikeness of functions $\lambda f(z) + (1-\lambda)z$, $0 \leq \lambda \leq 1$, where $f(z) \in N$ with $\text{Re}\{f(z)/z\} > \frac{1}{2}$ in Δ.

The former problem was first considered by MacGregor [45] who solved for the cases $g(z) \in S^*$, $p(z) \in P$ and $g(z) \in S^c$, $p(z) \in P$. Since published in 1963, these results by MacGregor have been further extended and various other cases have been proved by, for example, Krzyż and Reade [36], Ratti [69], Causey and Merkes [14], Shaffer [82], Tuan and Anh [92]. We shall approach this problem from a unified stand-point by considering

$p(z) \in P(A, B)$. The problem (ii) was first treated by Trimble [91] when $f(z) \in S^c$. We note that $f(z) \in S^c$ implies $\mathrm{Re}\{f(z)/z\} > \frac{1}{2}$ in Δ. A related problem is also considered when the identity function z in $\lambda f(z) + (1-\lambda)z$ is replaced by a function connected to f(z) through some integral operator.

1.2 The functional $\mathrm{Re}\{\alpha\, p(z) + \beta z p'(z)/p(z)\}$, $\alpha \geq 0$, $\beta \geq 0$, over $P(A, B)$

From the definition of $P(A, B)$ we have that

$$p(z) \prec \frac{1 + Az}{1 + Bz} \quad , \quad z \in \Delta ,$$

for every $p(z) \in P(A, B)$. Thus, an application of the Subordination Principle (see 0.2.5) yields that the image of $|z| \leq r$ under every $p(z) \in P(A, B)$ is contained in the disc

(1.2.1) $$|p(z) - a| \leq d ,$$

where

(1.2.2) $$a = \frac{1 - ABr^2}{1 - B^2r^2} \quad , \quad d = \frac{(A-B)r}{1 - B^2r^2} .$$

From (1.2.1) and (1.2.2), it follows immediately that if $p(z) \in P(A, B)$, then, on $|z| = r < 1$,

(1.2.3) $$\frac{1 - Ar}{1 - Br} \leq \mathrm{Re}\{p(z)\} \leq |p(z)| \leq \frac{1 + Ar}{1 + Br} .$$

The bounds are attained for the function $p(z) = (1+Az)/(1+Bz)$.

The basic tool which we rely on to handle extremal problems for $P(A, B)$ is the following inequality derived by Dieudonné in his work [18] on bounded functions.

1.2.1 <u>Lemma</u>. *If* $w(z) \in \mathcal{B}$, *then for* $|z| < 1$,

$$(1.2.4) \qquad |zw'(z) - w(z)| \leq \frac{|z|^2 - |w(z)|^2}{1 - |z|^2} .$$

<u>Proof</u>. Write $w(z) = z\psi(z)$, where $\psi(z)$ is regular and satisfying $|\psi(z)| \leq 1$ in Δ. From the well-known result due to Carathéodory that

$$(1.2.5) \qquad |\psi'(z)| \leq \frac{1 - |\psi(z)|^2}{1 - |z|^2} , \quad z \in \Delta ,$$

(see Carathéodory [13, p.18]) the assertion follows and is easily seen to be sharp for functions of the form $w(z) = z(z-c)/(1-cz)$, ~~$|c| \leq 1$~~.
where c is any complex number such that $|c| \leq 1$.

From now on, we shall refer to Lemma 1.2.1 as Dieudonné's lemma. We now prove

1.2.2 <u>Theorem</u>. *If* $p(z) \in P(A, B)$, $\alpha \geq 0$, $\beta \geq 0$, *then on* $|z| = r < 1$,

$$\mathrm{Re}\{\alpha\, p(z) + \beta\frac{zp'(z)}{p(z)}\} \geq \begin{cases} \dfrac{\alpha - [\beta(A-B) + 2\alpha A]r + \alpha A^2r^2}{(1-Ar)(1-Br)} , & R_1 \leq R_2 , \\[2ex] \beta\dfrac{A + B}{A - B} + \dfrac{2}{(A-B)(1-r^2)} [(L_1K_1)^{\frac{1}{2}} - \beta(1-ABr^2)] , & R_2 < R_1 , \end{cases}$$

where $R_1 = (L_1/K_1)^{\frac{1}{2}}$, $R_2 = (1-Ar)/(1-Br)$, $L_1 = \beta(1-A)(1+Ar^2)$,
$K_1 = \alpha(A-B)(1-r^2) + \beta(1-B)(1+Br^2)$. *The result is sharp.*

<u>Proof</u>. From the representation (0.1.7) of $p(z)$ we deduce easily that

$$\alpha\, p(z) + \beta\,\frac{zp'(z)}{p(z)} = \alpha\,\frac{1+Aw(z)}{1+Bw(z)} + \beta(A-B)\,\frac{zw'(z)}{|1+Aw(z)||1+Bw(z)|}\ ,$$

for some $w(z) \in \mathcal{B}$, $z \in \Delta$. Applying Dieudonné's lemma to the second term of the right hand side, we find

$$(1.2.6)\qquad \mathrm{Re}\{\alpha\, p(z) + \beta\,\frac{zp'(z)}{p(z)}\} \geq \mathrm{Re}\{\alpha\,\frac{1+Aw(z)}{1+Bw(z)} + \beta\,\frac{(A-B)\,w(z)}{[1+Aw(z)][1+Bw(z)]}\}$$

$$- \beta(A-B)\,\frac{|z|^2 - |w(z)|^2}{(1-|z|^2)|1+Aw(z)||1+Bw(z)|}\ .$$

From (0.1.7) we also have, for any $w(z) \in \mathcal{B}$,

$$w(z) = \frac{p(z) - 1}{A - B\,p(z)}\ ,\quad p(z) \in P(A,\ B)\ ,\quad z \in \Delta\ .$$

Hence, in terms of $p(z)$, inequality (1.2.6) becomes, on $|z| = r$,

$$(1.2.7)\qquad \mathrm{Re}\{\alpha\, p(z) + \beta\frac{zp'(z)}{p(z)}\} \geq \beta\,\frac{A+B}{A-B} + \frac{1}{A-B}\,\mathrm{Re}\{[\alpha(A-B)-\beta B]p(z) - \frac{\beta A}{p(z)}\}$$

$$- \beta\,\frac{r^2|B\,p(z)-A|^2 - |1-p(z)|^2}{(A-B)(1-r^2)|p(z)|}\ .$$

We recall that the image of $|z| \leq r$ under $p(z)$ is contained in the disc $|p(z) - a| \leq d$, where a, d are as given by (1.2.2). Now, put $p(z) = a + u + iv$, $|p(z)| = R$ and denote the right-hand side of (1.2.7) by $S(u,\ v)$, then

$$S(u,\ v) = \beta\frac{A+B}{A-B} + \frac{1}{A-B}\,\{[\alpha(A-B)-\beta B](a+u) - \beta\frac{A(a+u)}{R^2} - \beta\frac{1-B^2r^2}{1-r^2}\cdot\frac{d^2-u^2-v^2}{R}\}\ .$$

This yields

$$(1.2.8) \qquad \frac{\partial S}{\partial v} = \frac{\beta}{A - B} \cdot \frac{v}{R^4} \cdot T(u, v)$$

where

$$T(u, v) = 2 A (a+u) + \frac{1-B^2r^2}{1-r^2} [2 R^3 + (d^2-u^2-v^2)R]$$

$$\geq 2(a+u) [A + \frac{1-B^2r^2}{1-r^2} (a - d)^2]$$

as $d^2-u^2-v^2 \geq 0$ and $R^3 \geq (a+u)(a-d)^2$. Now

$$(1.2.9) \quad A + \frac{1-B^2r^2}{1-r^2} (a-d)^2 \geq A + (a-d)^2 = \frac{(1+B)(1-Ar)^2 + (A-B)(1-ABr^2)}{(1-Br)^2} > 0.$$

Hence $T(u, v) > 0$ and it follows from (1.2.8) that the minimum of $S(u, v)$ on the disc $|p(z) - a| \leq d$ is attained when $v = 0$ and $u \in [-d, d]$. Setting $v = 0$ in the expression for $S(u, v)$ we get

$$S(u, 0) = \beta\frac{A+B}{A-B} + \frac{1}{A-B} \{\beta\frac{(1-A)(1+Ar^2)}{1-r^2} \cdot \frac{1}{a+u} + \frac{\alpha(A-B)(1-r^2)+\beta(1-B)(1+Br^2)}{1 - r^2} \cdot (a+u)$$

$$- 2\beta \frac{1-ABr^2}{1-r^2} \} .$$

From

$$\frac{d S(u, 0)}{du} = \frac{1}{A-B} [- \frac{\beta(1-A)(1+Ar^2)}{1-r^2} \cdot \frac{1}{(a+u)^2} + \frac{\alpha(A-B)(1-r^2)+\beta(1-B)(1+Br^2)}{1 - r^2}],$$

we see that the absolute minimum of $S(u, 0)$ occurs at the point $u_0 = (L_1/K_1)^{\frac{1}{2}} - a$ if u_0 lies inside $[-d, d]$, its value being

$$S(u_0, 0) = \beta\frac{A+B}{A-B} + \frac{2}{(A-B)(1-r^2)} [(L_1K_1)^{\frac{1}{2}} - \beta(1-ABr^2)] .$$

Now, from the conditions $-1 \le B < A \le 1$, $\alpha \ge 0$, $\beta \ge 0, r < 1$, it is clear that

$$(1.2.10) \qquad (a+u_0)^2 \le \frac{(1-A)(1+Ar^2)}{(1-B)(1+Br^2)} < \frac{1+Ar^2}{1+Br^2} < \frac{1+Ar}{1+Br} < \left(\frac{1+Ar}{1+Br}\right)^2 = (a+d)^2 \ .$$

Thus $u_0 < d$. However, u_0 is not always greater than $-d$. For the case $u_0 \le -d$, that is, if $R_1 \le R_2$, the absolute minimum of $S(u, 0)$ occurs at the end-point $u = -d$, the value of which is

$$S(-d, 0) = \frac{\alpha-[\beta(A-B)+2\alpha A]r + \alpha A^2r^2}{(1-Ar)(1-Br)} \quad .$$

To see that the bounds are sharp, we consider the functions

$$p_1(z) = \frac{1 + Az}{1 + Bz} \quad \text{for} \quad R_1 \le R_2$$

and

$$p_2(z) = \frac{1 + Aw_1(z)}{1 + Bw_1(z)} \quad \text{for} \quad R_2 \le R_1 \quad ,$$

where $w_1(z) = z(z - c_1)/(1 - c_1 z)$ with c_1 being determined by the condition $\mathrm{Re}\{[1 + Aw_1(z)]/[1 + Bw_1(z)]\} = R_1$ at $z = -r$. It may be verified that $|c_1| \le 1$. In fact, from the above condition and the inequalities

$$R_2 \le R_1 \le a + d \ ,$$

we have

$$\frac{1 - Ar}{1 - Br} \le \frac{1 + Ax}{1 + Bx} \le \frac{1 + Ar}{1 + Br} \quad , \quad x = w_1(-r) \quad .$$

Hence $|x| \le r$, and so $x^2 \le r^2$, which yields

$$\frac{r^2(r + c_1)^2}{(1 + c_1r)^2} \le r^2 \quad ,$$

that is, $|c_1| \leq 1$. Since Dieudonné's lemma is sharp for functions of the form $z(z - c)/(1 - cz)$, $|c| \leq 1$, the proof of the theorem is thus completed.

1.2.3 Corollary. *Let* $p(z) \in P$, $0 \leq \gamma < 1$; *then on* $|z| = r < 1$,

$$(1.2.11)\quad \mathrm{Re}\{\frac{zp'(z)}{p(z)+\gamma/(1-\gamma)}\} \geq \begin{cases} -\dfrac{2(1-\gamma)r}{[1+(2\gamma-1)r](1+r)}, & R_3 \leq R_4, \\ -\dfrac{\gamma}{1-\gamma} + \dfrac{1}{1-\gamma}\left[2\left(\dfrac{\gamma+\gamma(1-2\gamma)r^2}{1-r^2}\right)^{\frac{1}{2}} - \dfrac{1+(1-2\gamma)r^2}{1-r^2}\right], & R_4 \leq R_3, \end{cases}$$

where $R_3 = \{[\gamma+\gamma(1-2\gamma)r^2]/(1-r^2)\}^{\frac{1}{2}}$, $R_4 = [1-(1-2\gamma)r]/(1+r)$.

Proof. Put $q(z) = \gamma + (1-\gamma)\, p(z)$; then $q(z) \in P(1 - 2\gamma, -1)$ and

$$\frac{z\, q'(z)}{q(z)} = \frac{zp'(z)}{p(z) + \gamma/(1-\gamma)} .$$

The result now follows from Theorem 1.2.2 with $A = 1 - 2\gamma$, $B = -1$, $\alpha = 0$, $\beta = 1$.

1.2.4 Remark. The functions which give equality in (1.2.11) may be derived from the extremal functions of Theorem 1.2.2. However, in this simple case with $p(z) \in P$, we may use Robertson and Zmorovič's results to determine them in an alternative form. Indeed, Robertson [75] showed that these functions must be of the form

$$p(z) = \frac{1+\lambda}{2} \cdot \frac{1+ze^{i\theta}}{1-ze^{i\theta}} + \frac{1-\lambda}{2} \cdot \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} ,$$

where $-1 \leq \lambda \leq 1$, $0 \leq \theta \leq 2\pi$. Taking into account (0.1.15) and the fact that the minimum of $S(u, v)$ is attained at a point on the diameter $v = 0$, we may put $\lambda = 0$. Hence an extremal function for the case $R_4 \le R_3$ will be

$$p_0(z) = \tfrac{1}{2} \cdot \frac{1 + ze^{i\theta}}{1 - ze^{i\theta}} + \tfrac{1}{2} \cdot \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} ,$$

where θ is determined from the condition

$$\mathrm{Re}\{\gamma + (1-\gamma)p_0(z)\} = R_3$$

at $z = -r$, or equivalently, the equation

$$\frac{1 + 2\gamma r\cos\theta - (1-2\gamma)r^2}{1 + 2r\cos\theta + r^2} = R_3 \quad . \tag{1.2.12}$$

For the case $R_3 \le R_4$, equality in (1.2.11) occurs for the function $p(z) = (1 + z)/(1 - z)$.

Putting $\gamma = 0$ in (1.2.11) we obtain, for $p(z) \in \mathsf{P}$,

$$\mathrm{Re}\{\frac{zp'(z)}{p(z)}\} \ge -\frac{2r}{1 - r^2} \quad , \quad |z| = r < 1 \ . \tag{1.2.13}$$

This inequality was derived previously by Libera [38] using Robertson's method.

1.3 A problem of MacGregor concerning close-to-starlike functions

In this section we shall be concerned with the problem of determining the radius of starlikeness of functions $f(z) \in \mathsf{N}$ which satisfy the condition

$$\frac{f(z)}{g(z)} = p(z) \quad , \quad z \in \Delta \quad ,$$

where g(z) belongs to some subclass of S or N and $p(z) \in P(A, B)$. We shall also briefly discuss the similar problem of determining the radius of convexity of functions $f(z) \in N$ for which

$$\frac{f'(z)}{g'(z)} = p(z) \quad , \quad z \in \Delta \quad ,$$

where g(z), p(z) are as above.

We denote by P_α the class of functions $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ with real part greater than α, $0 \le \alpha < 1$, in Δ. It is clear that $[p(z) - \alpha]/(1-\alpha) \in P$. Hence

(1.3.1) $$p(z) = \alpha + (1-\alpha)\, q(z) \,,$$

for some $q(z) \in P$.

Let $f(z) \in N$ be such that $\mathrm{Re}\{f(z)/z\} > 0$ in Δ. It is known (see Polya and Szegö [66, problem 3]) that f(z) is univalent in $|z| < \sqrt{2} - 1$. MacGregor [45] showed that this function is also starlike in $|z| < \sqrt{2} - 1$. Since every function f(z) in S^c satisfies the condition $\mathrm{Re}\{f(z)/z\} > \frac{1}{2}$ in Δ as established by Strohhäcker [88] and Marx [49], MacGregor [44] considered the class

$$R_{\frac{1}{2}} = \{f(z) \in N \; ; \; \mathrm{Re}\{f(z)/z\} > \tfrac{1}{2} \,, \quad z \in \Delta\}$$

and proved that every $f(z) \in R_{\frac{1}{2}}$ is starlike in $|z| < 1/\sqrt{2}$.

It is natural to generalise $R_{\frac{1}{2}}$ to the class (see Yamaguchi, Proc. Amer. Math. Soc. 17 (1966), 588-591)

$$R_\alpha = \{f(z) \in N \; ; \; \mathrm{Re}\{f(z)/z\} > \alpha \; , \; 0 \le \alpha < 1 \; , \; z \in \Delta \}$$

and ask what the radius of starlikeness of R_α is. This is given in the following

1.3.1 Theorem [92]. *The radius of starlikeness* σ_1 *of* R_α *is given by*

$$\sigma_1 = \begin{cases} \left[\dfrac{2(1-\alpha)}{1-2\alpha}\right]^{\frac{1}{2}} - 1 & , \; 0 \le \alpha \le 1/10 \; , \\ \left\{\dfrac{\alpha}{\alpha + [\alpha(1-\alpha)]^{\frac{1}{2}}}\right\}^{\frac{1}{2}} & , \; 1/10 \le \alpha < 1. \end{cases}$$

Proof. Since $f(z)/z \in P_\alpha$, we may write by (1.3.1) that

$$\frac{f(z)}{z} = \alpha + (1-\alpha)\, p(z) \quad ,$$

for some $p(z) \in P$. From this representation, we deduce

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z) + \alpha/(1-\alpha)} \quad .$$

Making use of (1.2.11), we find on $|z| = r$,

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} \ge \begin{cases} 1 - \dfrac{2(1-\alpha)}{[1+(2\alpha-1)r](1+r)} \; , & R_3 \le R_4 \; , \\ 1 - \dfrac{\alpha}{1-\alpha} + \dfrac{1}{1-\alpha}\left[2\left(\dfrac{\alpha+\alpha(1-2\alpha)r^2}{1-r^2}\right)^{\frac{1}{2}} - \dfrac{1+(1-2\alpha)r^2}{1-r^2}\right], & R_4 \le R_3 \; , \end{cases}$$

where R_3, R_4 are as given in Corollary 1.2.3. Consequently, for $R_3 \le R_4$, the radius of starlikeness σ_1 of $f(z)$ is given by the smallest root in (0, 1] of the equation

$$(2\alpha-1)r^2 + 2(2\alpha-1)r + 1 = 0 \; ,$$

which is $[2(1-\alpha)/(1-2\alpha)]^{\frac{1}{2}} - 1$. Clearly this value is not real if $\alpha > \frac{1}{2}$. Furthermore, since $\sigma_1 \le 1$, we must have $\alpha \le 1/3$. For $R_4 \le R_3$, σ_1 is given by the smallest root in $(0, 1]$ of the equation

$$(1-2\alpha)r^4 + 2\alpha r^2 - \alpha = 0 ,$$

which is $\{\alpha/[\alpha+(\alpha(1-\alpha))^{\frac{1}{2}}]\}^{\frac{1}{2}}$. The value of α which determines the transition from the first case to the second one is given by

$$\left[\frac{2(1-\alpha)}{1-2\alpha}\right]^{\frac{1}{2}} - 1 = \left\{\frac{\alpha}{\alpha + [\alpha(1-\alpha)]^{\frac{1}{2}}}\right\}^{\frac{1}{2}} ,$$

that is , $\alpha = 1/10$.

In view of Remark 1.2.4, we deduce that the result is sharp for

$$f(z) = \alpha z + (1-\alpha)z.\frac{1+z}{1-z} \qquad , \text{ for } 0 \le \alpha \le 1/10 ,$$

$$f(z) = \alpha z + \tfrac{1}{2}(1-\alpha)z \left(\frac{1+ze^{i\theta}}{1-ze^{i\theta}} + \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right) , \text{ for } 1/10 \le \alpha < 1 ,$$

where θ satisfies equation (1.2.12) with $r = \sigma_1$, γ replaced by α .

1.3.2 Remark. In as early as 1934, Wolff [96] showed that if $f(z)$ is regular and satisfies $\text{Re}\{f'(z)\} > 0$ in $\text{Re } z > 0$, then it is univalent there. Noshiro [61] and Warschawski [94] each independently demonstrated that the condition $\text{Re}\{f'(z)\} > 0$ is sufficient for the univalence of $f(z)$ in any convex domain. Concerning the convexity of such $f(z)$, MacGregor [43] proved that each function $f(z) \in \mathbb{N}$ with $\text{Re}\{f'(z)\} > 0$ in Δ is convex in $|z| < \sqrt{2} - 1$. Hallenbeck [29] improved this radius of convexity of $f(z)$ to $1/\sqrt{2}$ when the condition $\text{Re}\{f'(z)\} > 0$ is replaced by $\text{Re}\{f'(z)\} > \frac{1}{2}$ in Δ and put forward the problem of determining the radius of convexity of functions $f(z) \in \mathcal{N}$ which satisfy the more general condition that $\mathrm{Re}\{f'(z)\} > \alpha$ for arbitrary α in $[0, 1)$, $z \in \Delta$. This problem is readily solved using Theorem 1.3.1 as follows.

Let $f(z) \in \mathcal{N}$ be such that $\mathrm{Re}\{f'(z)\} > \alpha$ in Δ and define $g(z) = zf'(z)$, $z \in \Delta$. Then it follows easily that $g(z)$ is starlike in $|z| < r$ if and only if $f(z)$ is convex there. Furthermore,

$$\mathrm{Re}\{\frac{g(z)}{z}\} = \mathrm{Re}\{f'(z)\} > \alpha \ , \ z \in \Delta \ .$$

Hence the radius of convexity of $f(z)$ is σ_1 as given by Theorem 1.3.1.

In the cited paper [45], MacGregor considered a more general problem : What is the radius of starlikeness of $f(z) \in \mathcal{N}$ for which $\mathrm{Re}\{f(z)/g(z)\} > 0$ in Δ, where $g(z) \in \mathcal{S}$?

MacGregor solved this problem for the cases $g(z) \in \mathcal{S}^*$ and $g(z) \in \mathcal{S}^c$. It was Krzyż and Reade [36] who settled the problem for $g(z) \in \mathcal{S}$. Again, the results $\mathcal{S}^c \subseteq \mathcal{R}_{\frac{1}{2}}$ and $\mathcal{S}^c \subseteq \mathcal{S}^*_{\frac{1}{2}}$ motivate the consideration of the cases $g(z) \in \mathcal{R}_{\frac{1}{2}}$ and $g(z) \in \mathcal{S}^*_{\alpha}$ (see Ratti [69]) .

These authors also investigated the problem when the condition $\mathrm{Re}\{f(z)/g(z)\} > 0$ in Δ is replaced by a more restrictive condition that $|f(z)/g(z) - 1| < 1$ in Δ . Shaffer [82] generalised both this and the former condition by looking at the functions $f(z) \in \mathcal{N}$ which satisfy

$$\left|\frac{f(z)}{g(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \quad , \quad 0 \le \alpha < 1 \ . \tag{1.3.2}$$

Her investigation is based on the result [82] that if $p(z) = 1 + p_1 z + \ldots$ is regular and satisfies $|p(z) - 1/2\alpha| < 1/2\alpha$ in Δ , then on $|z| = r < 1$,

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{2(1-\alpha)r}{[1+(1-2\alpha)r](1-r)} .$$

However, as mentioned by Shaffer in [82] , this approach does not lead to sharp estimates, except for $\alpha = 0$, when the classes considered are characterised by the condition

(1.3.3) $$\mathrm{Re}\{\frac{f(z)}{g(z)}\} > \alpha , \quad 0 \le \alpha < 1 , \quad z \in \Delta ,$$

instead of (1.3.2). Sharp results for these problems have been established by Tuan and Anh in [92].

With Theorem 1.2.2 available, we may, in fact, establish best possible results for functions which satisfy a much more general condition that

(1.3.4) $$\frac{f(z)}{g(z)} = p(z) \quad , \quad p(z) \in \mathcal{P}(A, B) , \; z \in \Delta .$$

This comprises (1.3.2) and (1.3.3) as special cases. Indeed, from (1.3.3) we put

(1.3.5) $$\frac{f(z)}{g(z)} = \alpha + (1-\alpha)\, p(z) \quad , \; p(z) \in \mathcal{P} .$$

Then, since every $p(z) \in \mathcal{P}$ may be written as $p(z) = [1+w(z)]/[1-w(z)]$, $w(z) \in \mathcal{B}$, (1.3.5) becomes

$$\frac{f(z)}{g(z)} = \frac{1 + (1-2\alpha)w(z)}{1 - w(z)} .$$

Thus, putting $A = 1-2\alpha$, $B = -1$, (1.3.4) reduces to (1.3.3). For (1.3.2), we put

$$\psi(z) = 2\alpha\left[\frac{f(z)}{g(z)} - \frac{1}{2\alpha}\right] .$$

Then $|\psi(z)| < 1$ and $\psi(0) = 2\alpha - 1 = B$. Now let $w(z) = [\psi(z)-B]/[1-B\psi(z)]$, then $w(z) \in \mathcal{B}$ and an easy calculation yields

$$\frac{f(z)}{g(z)} = \frac{1 + w(z)}{1 + (2\alpha-1)w(z)} .$$

Consequently, Shaffer's condition (1.3.2) corresponds to (1.3.4) with $A = 1$, $B = 2\alpha - 1$.

We have yet to mention two other classes whose characterising conditions are also covered by (1.3.4), namely

$$S^{c*}_{\beta,\alpha} = \{f(z) \in N ; \left|\frac{f(z)}{g(z)} - 1\right| / \left|\frac{f(z)}{g(z)} + 1\right| < \beta \text{ , } 0 < \beta \le 1, \; g(z) \in S^*_\alpha \text{ , } z \in \Delta\} \text{ ,}$$

$$S^{c*}_{\beta\gamma,\alpha} = \{f(z) \in N ; \left|\frac{f(z)}{g(z)} - \beta\right| < \gamma, \; 0 < \gamma \le \beta, \; g(z) \in S^*_\alpha \text{ , } z \in \Delta\} \text{ .}$$

The class $S^{c*}_{\beta\gamma,\alpha}$, introduced by Tuan and Anh in [93] , reduces to those defined by the conditions (1.3.2) and (1.3.3) by suitable choices of β,γ . As before, we can easily show that the conditions $|f(z)/g(z)-1|/|f(z)/g(z)+1|<\beta$ and $|f(z)/g(z)-\beta| < \gamma$ are equivalent to (1.3.4) with $A = \beta$, $B = -\beta$ and $A = \gamma + \beta B$, $B = (1-\beta)/\gamma$ respectively.

In the light of these observations we see that various results on MacGregor's problem may be unified and generalised by considering solely the class which is characterised by condition (1.3.4).

We present the following result which covers MacGregor [45, Theorems 3,4], [46, Theorems 3,4] , Ratti [69, Theorems 3,6] , Shaffer [82, Theorems 2,4], Tuan and Anh [92, Theorem 3.3] , [93, Theorem 3] , by appropriate choices of α and A, B as determined above.

1.3.3 <u>Theorem</u>. *Let* $f(z) \in \mathsf{N}$ *be such that* $f(z)/g(z) \in P(A, B)$, *where* $g(z) \in S_\alpha^*$. *Then the radius of starlikeness of* $f(z)$ *is given by the smallest root in* $(0, 1]$ *of*

$$\text{(i)}\quad AB(1-2\alpha)r^3 - (AB+2B-2\alpha A-2\alpha B)r^2 + (1-2\alpha+2A)r - 1 = 0\ ,\ \textit{for}\ R_1' \le R_2\ ,$$

$$\text{(ii)}\quad [\alpha^2(A-B) - B(1-2\alpha)(1-A)]r^4 + 2(1-\alpha)(B+\alpha A-\alpha B-AB)r^3$$

$$+ [(1-A)(2\alpha-1-B) + (1-\alpha)^2(A-B)]r^2 + 2(1-A)(1-\alpha)r -1+A = 0,\quad \textit{for}\ R_2 \le R_1'\ ,$$

where

$$R_1' = \left[\frac{(1-A)(1+Ar^2)}{(1-B)(1+Br^2)}\right]^{\frac{1}{2}}\ ,\quad R_2 = \frac{1 - Ar}{1 - Br}\ .$$

<u>Proof</u>. For $f(z)$ as given, we deduce that

$$(1.3.6)\qquad \frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)}\ ,\qquad z \in \Delta\ .$$

Since $g(z) \in S_\alpha^*$, we have $zg'(z)/g(z) \in P_\alpha$. Thus in view of (1.2.1) and (1.2.2) ,

$$(1.3.7)\qquad \mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \ge \frac{1 - (1-2\alpha)r}{1 + r}\ ,\qquad |z| = r < 1\ .$$

The required equations giving the radius of starlikeness of $f(z)$ may now be derived from (1.3.6), (1.3.7) and Theorem 1.2.2 with $\alpha = 0$, $\beta = 1$.

The result is sharp for

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \cdot \frac{1 + Az}{1 + Bz} \quad , \quad \text{for } R_1' \le R_2 \ ,$$

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \cdot \frac{1+Aw_2(z)}{1+Bw_2(z)} \, , \quad \text{for } R_2 \le R_1' \quad ,$$

where $w_2(z) = z(z-c_2)/(1-c_2z)$ with c_2 being determined by the condition $\mathrm{Re}\{[1 + Aw_2(z)]/[1 + Bw_2(z)]\} = R_1'$ at $z = -r$.

As mentioned earlier , Ratti [69] determined the radius of starlikeness of functions $f(z) \in \mathbb{N}$ for which $\mathrm{Re}\{f(z)/g(z)\} > 0$ in Δ , where $g(z) \in \mathbb{N}$ and satisfies

$$\text{(1.3.8)} \qquad \mathrm{Re}\{\frac{g(z)}{z}\} > \tfrac{1}{2} \, , \quad z \in \Delta \quad .$$

Causey and Merkes [14] generalised this result by replacing the identity function z in condition (1.3.8) by a function $s(z) \in S_\alpha^*$. Their analysis relies on the result that for $p(z) \in P_{\frac{1}{2}}$ and $|z| = r < 1$,

$$\mathrm{Re}\{\frac{zp'(z)}{p(z)}\} \ge \begin{cases} -\dfrac{r}{1 + r} & , \ 0 < r \le 1/3 \ , \\[2ex] \dfrac{r^2 + 2\sqrt{2}\,(1-r^2)^{\frac{1}{2}}-3}{1 - r^2} & , \ 1/3 \le r \le (8\sqrt{2} - 11)^{\frac{1}{2}} \cong 0.56 \ . \end{cases}$$

We now know that the second bound is best possible over the complete range $1/3 \le r < 1$ as shown by Corollary 1.2.3 with $\gamma = \frac{1}{2}$. Also, in view of Corollary 1.2.3, we may extend Causey and Merkes' result by considering the condition

$$\text{Re}\{\frac{g(z)}{s(z)}\} > \beta \ , \ z \in \Delta \ ,$$

where β is arbitrary in $[0, 1)$ and $s(z) \in S_\alpha^*$. We prove

1.3.4 <u>Theorem</u>. *Let* $f(z) \in N$ *be such that* $\text{Re}\{f(z)/g(z)\} > 0$ *in* Δ *, where* $g(z)$ *satisfies* $\text{Re}\{g(z)/s(z)\} > \beta$ *in* Δ *,* $0 \le \beta < 1$ *and* $s(z) \in S_\alpha^*$. *Then the radius of starlikeness* σ_2 *of* $f(z)$ *is given by the smallest root in* $(0, 1]$ *of*

(i) $(1-2\alpha)(2\beta-1)r^3 + (4\alpha\beta-4\alpha-10\beta+7)r^2 + (2\alpha+4\beta-7)r + 1 = 0,$ *for* $0 \le \beta \le \beta_0$,

(ii) $[(1-2\alpha)\beta + \alpha^2(1-\beta)]r^4 - 2(2-\alpha)(\beta-\alpha+\alpha\beta)r^3 + [(2-\alpha)^2(1-\beta) + 2\alpha\beta]r^2$

$+ 2\beta(2-\alpha)r - \beta = 0$ *, for* $\beta_0 \le \beta < 1$ *, some* $\beta_0 \in (0, 1)$.

<u>Proof</u>. For $f(z)$ as defined, there exist $p(z) \in P$ and $q(z) \in P$ such that $f(z)/g(z) = q(z)$ and $g(z)/s(z) = \beta + (1-\beta)p(z)$, $z \in \Delta$. From these representations we deduce

$$\frac{zf'(z)}{f(z)} = \frac{zs'(z)}{s(z)} + \frac{zq'(z)}{q(z)} + \frac{zp'(z)}{p(z) + \beta/(1-\beta)} \ . \tag{1.3.9}$$

As shown in the proof of Theorem 1.3.3 ,

$$\text{Re}\{\frac{zs'(z)}{s(z)}\} \ge \frac{1 - (1-2\alpha)r}{1 + r} \ , \quad |z| = r \, . \tag{1.3.10}$$

Applying the results (1.3.10), (1.2.13) and (1.2.11) to the terms of the right-hand side of equation (1.3.9) respectively we obtain the equations giving the radius of starlikeness of $f(z)$ to be

$$F(r) \equiv (1-2\alpha)(2\beta-1)r^3 + (4\alpha\beta-4\alpha-10\beta+7)r^2 + (2\alpha+4\beta-7)r + 1 = 0$$

for $R_3 \le R_4$, and

$$G(r) \equiv [(1-2\alpha)\beta + \alpha^2(1-\beta)]r^4 - 2(2-\alpha)(\beta-\alpha+\alpha\beta)r^3 + [(2-\alpha)^2(1-\beta) + 2\alpha\beta]r^2$$
$$+ 2\beta(2-\alpha)r - \beta = 0$$

for $R_4 \le R_3$, where R_3, R_4 are as given in Corollary 1.2.3. We note that $F(0) = 1$, $F(1) = -4\beta < 0$. Thus $F(r)$ has a root in $(0, 1)$; we denote its smallest root in $(0, 1)$ by r_1. Similarly, $G(r)$ has a root in $(0, 1)$ as $G(0) = -\beta < 0$ and $G(1) = 4(1-\beta) > 0$; its smallest root in $(0, 1)$ is denoted by r_2. Then β_0 is determined from the equation $r_1 = r_2$.

Equality in (1.3.10) is attained by the function $s(z) = z/(1-z)^{2(1-\alpha)}$, while that for (1.2.13) is reached by the function $q(z) = (1+z)/(1-z)$. Thus, together with Remark 1.2.4, we find that the result is sharp for

$$f(z) = \frac{z(1+z)}{(1-z)^{3-2\alpha}} \cdot \frac{1 + (1-2\beta)z}{1 - z} , \qquad \text{for } 0 \le \beta \le \beta_0 ,$$

$$f(z) = \frac{z(1+z)}{(1-z)^{3-2\alpha}} \cdot [\beta+\tfrac{1}{2}(1-\beta)(\frac{1+ze^{i\theta}}{1-ze^{i\theta}} + \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}})] , \text{ for } \beta_0 \le \beta < 1 ,$$

where θ satisfies equation (1.2.12) with $r = \sigma_2$ and γ replaced by β.

Putting $\beta = 0, \frac{1}{2}$ we obtain Theorems 3.1 and 3.2 of Causey and Merkes [14] respectively.

1.3.5 Remark. In Remark 1.3.2, we have looked at a special case of the problem of the radii of convexity of subclasses of close-to-convex functions. The analysis shows that this problem may be approached in a similar fashion as that for the radii of starlikeness of subclasses of close-to-starlike functions. To give a further example, we determine the radius of convexity of $f(z) \in N$ for which $\mathrm{Re}\{f'(z)/g'(z)\} > 0$ in Δ, where $g(z) \in S_\alpha^*$. In other words, $f'(z) = g'(z)p(z)$, for some $p(z) \in P$. This representation yields

$$(1.3.11) \qquad 1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zg''(z)}{g'(z)} + \frac{zp'(z)}{p(z)} , \quad z \in \Delta .$$

Again, for $g(z) \in S_\alpha^*$, we may write

$$\frac{zg'(z)}{g(z)} = \alpha + (1-\alpha)q(z) , \; q(z) \in P , \; z \in \Delta ,$$

which implies

$$1 + \frac{zg''(z)}{g'(z)} = \frac{zg'(z)}{g(z)} + \frac{zq'(z)}{q(z) + \alpha/(1-\alpha)} .$$

Hence, together with (1.3.11), we get

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)} + \frac{zq'(z)}{q(z) + \alpha/(1-\alpha)} , \quad z \in \Delta .$$

The right-hand side is almost the same as (1.3.9). Thus, with similar argument, we deduce that the radius of convexity of $f(z)$ is given by the smallest root in $(0, 1]$ of

(i) $(2\alpha-1)^2r^3 - (7-14\alpha+4\alpha^2)r^2 + (7-6\alpha)r - 1 = 0$, for $0 \le \alpha \le \alpha_0$,

(ii) $(8\alpha^2-3\alpha-1)r^4 + 4(3\alpha-1)r^3 + 2(3\alpha-4\alpha^2-3)r^2 - 4(1+\alpha)r + 5\alpha - 1 = 0$, for $\alpha_0 \le \alpha < 1$, where α_0 is determined by equating these two roots.

Theorems 4 and 5 of Ratti [70] correspond to the cases $\alpha = 0$ and $\alpha = \frac{1}{2}$ respectively.

1.4 A problem of Trimble

In [30, Problem 6.11], Hayman posed the following question:

If $f(z), g(z) \in S^c$ and $0 < \lambda < 1$, is $h(z) = \lambda f(z) + (1-\lambda)g(z)$ in S^* ? MacGregor [48] showed that $h(z)$ need not even be in S. In fact, he proved that the largest disc in which every $h(z)$ is univalent has a radius of $1/\sqrt{2}$. This leads to the question : When is $h(z)$ starlike if $f(z)$ is in S^c but $g(z)$ is allowed to range through some restricted family ?

Trimble [91] discussed this question for the special case $g(z) = z$. He showed that the function $\lambda f(z) + (1-\lambda)z$, $f(z) \in S^c$, is starlike of order $(3\lambda-2)/2(2-\lambda)$ if $\lambda \ge 2/3$ and is in S^{cc} if $\lambda < 2/3$.

In this section we investigate the starlikeness of $\lambda f(z) + (1-\lambda)z$, $0 < \lambda < 1$, when it is only known that $f(z) \in R_{\frac{1}{2}}$. Also the radius of convexity of $\lambda f(z) + (1-\lambda)g(z)$ is given, where $f(z) \in S^*_{\frac{1}{2}}$, $g(z) = \int_0^z f(\xi)d\xi \,/\, \xi$ and $0 < \lambda < 1$.

1.4.1 Theorem. *Let* $k(z) = \lambda f(z) + (1-\lambda)z$, *where* $f(z) \in R_{\frac{1}{2}}$, $0 < \lambda < 1$. *Then the radius of starlikeness of* $k(z)$ *is*

$$\rho_1 = \{[2 - \lambda - (2\lambda-\lambda^2)^{\frac{1}{2}}]/2(1-\lambda)\}^{\frac{1}{2}} .$$

Proof. Since $\mathrm{Re}\{f(z)/z\} > \frac{1}{2}$ in Δ , we write

$$\frac{f(z)}{z} = \tfrac{1}{2}(1+p(z)) \quad , \quad z \in \Delta$$

for some $p(z) \in P$. Then

$$\frac{zk'(z)}{k(z)} = 1 + \frac{zp'(z)}{p(z) + (2-\lambda)/\lambda} .$$

Put $q(z) = 1 - \lambda/2 + \lambda p(z)/2$, then $q(z) \in P(\lambda-1, -1)$ and

$$\frac{zp'(z)}{p(z) + (2-\lambda)/\lambda} = \frac{zq'(z)}{q(z)} .$$

Hence

$$\frac{zk'(z)}{k(z)} = 1 + \frac{zq'(z)}{q(z)} .$$

An application of Theorem 1.2.2 with $\alpha = 0$, $\beta = 1$, $A = \lambda - 1$, $B = -1$ to the right-hand side gives

$$(1.4.1)\quad \mathrm{Re}\{\frac{zk'(z)}{k(z)}\} \geq \begin{cases} \dfrac{1 + 2(1-\lambda)r + (1-\lambda)r^2}{[1 + (1-\lambda)r](1+r)} , & \text{for } R_1 \leq R_2 , \\[2ex] 1 + \dfrac{\lambda-2}{\lambda} + \dfrac{2}{\lambda(1-r^2)} \{[2(2-\lambda)(1+(\lambda-1)r^2)(1-r^2)]^{\frac{1}{2}} \\ \qquad - [1+(\lambda-1)r^2]\} , & \text{for } R_2 \leq R_1 . \end{cases}$$

In the case $R_2 \leq R_1$, we derive the equation giving the radius of starlikeness of $k(z)$ to be

$$F(r) \equiv 2(1-\lambda)r^4 - 2(2-\lambda)r^2 + 2 - \lambda = 0 .$$

The only zero in $(0, 1)$ of $F(r)$ is $\rho_1 = \{[2-\lambda-(2\lambda-\lambda^2)^{\frac{1}{2}}]/2(1-\lambda)\}^{\frac{1}{2}}$. For $R_1 \leq R_2$, the right-hand side of (1.4.1) is always positive. Hence the radius of starlikeness of $k(z)$ is ρ_1.

In view of Remark 1.2.4, the result is sharp for the function

$$k(z) = z[1 - \frac{\lambda}{2} + \frac{\lambda}{4}\left(\frac{1+ze^{i\theta}}{1-ze^{i\theta}} + \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right)] ,$$

where θ satisfies the equation

$$\frac{1 + (2-\lambda)\rho_1\cos\theta + (1-\lambda)\rho_1^2}{1 + 2\rho_1\cos\theta + \rho_1^2} = \left\{\frac{(2-\lambda)[1+(\lambda-1)\rho_1^2]}{2(1-\rho_1^2)}\right\}^{\frac{1}{2}} .$$

1.4.2 <u>Remark</u>. As mentioned by Trimble in [91] , the function $k(z) = \lambda f(z) + (1-\lambda)z$, $f(z) \in S^c$, need not be starlike if $\lambda < 2/3$. Recently, Chichra and Singh [16] showed that if certain additional restriction is imposed on $f(z)$, then $k(z)$ is starlike for all λ in $(0, 1)$. In particular, they proved that if $f(z) \in S^c$, then the function $F(z) = \lambda \int_0^z f(\xi)d\xi / \xi + (1-\lambda)z$ is in S^* for all $\lambda \in (0, 1)$. We remark that for $\lambda \geq 2/3$, $F(z)$ is not only starlike, but, in fact, convex of order $(3\lambda-2)/2(2-\lambda)$. This can be seen as follows.

For $F(z)$ as defined, we have

$$zF'(z) = \lambda f(z) + (1-\lambda)z .$$

In view of Trimble's result that $\lambda f(z) + (1-\lambda)z \in S^*_{(3\lambda-2)/2(2-\lambda)}$ for $\lambda \geq 2/3$ and the fact that $F(z)$ is convex of order α if and only if $zF'(z)$ is starlike of order α, the assertion follows.

Theorem 1.4.1 establishes the radius of starlikeness of the functions $h(z) = \lambda f(z) + (1-\lambda)g(z)$, where $f(z) \in R_{\frac{1}{2}}$ and $g(z)$ takes the special form

$g(z) = z$. In the next theorem, we examine the case $f(z) \in S^*_{\frac{1}{2}}$ and $g(z) = \int_0^z f(\xi)d\xi \,/\, \xi$. We note that $f(z) \in S^*_{\frac{1}{2}}$ implies $g(z) \in S^*_{\frac{1}{2}}$.

1.4.3 Theorem. *Let* $h(z) = \lambda f(z) + (1-\lambda)\int_0^z f(\xi)d\xi \,/\, \xi$, *where* $f(z) \in S^*_{\frac{1}{2}}$, $0 < \lambda < 1$. *Then*,

(i) $h(z)$ *is close-to-convex in* Δ ;

(ii) *the radius of convexity of* $h(z)$ *is*

$$\rho_2 = \begin{cases} \dfrac{1 - 8\lambda + 4\lambda^2}{1-6\lambda+2\lambda^2+[12\lambda^3(2-\lambda)]^{\frac{1}{2}}} , & 0 < \lambda \le 1 - \sqrt{3}/2 , \\[2ex] \dfrac{1 - 8\lambda + 4\lambda^2}{1-6\lambda+2\lambda^2-[12\lambda^3(2-\lambda)]^{\frac{1}{2}}} , & 1 - \sqrt{3}/2 \le \lambda < 1 . \end{cases}$$

Proof. For $h(z)$ as given, we have

$$h'(z) = \lambda f'(z) + (1-\lambda)\frac{f(z)}{z} .$$

Hence,

$$\frac{zh'(z)}{f(z)} = \lambda \frac{zf'(z)}{f(z)} + 1 - \lambda$$

$$= \frac{\lambda}{2}[1 + p(z)] + 1 - \lambda \tag{1.4.2}$$

for some $p(z) \in P$ as $f(z) \in S^*_{\frac{1}{2}}$. Consequently,

$$\mathrm{Re}\{\frac{zh'(z)}{f(z)}\} > \frac{2 - \lambda}{2} > 0 , \quad z \in \Delta ,$$

which means that $h(z)$ is close-to-convex with respect to $f(z)$ in Δ (see (0.2.7)). Also, from (1.4.2) we deduce that

$$1 + \frac{zh''(z)}{h'(z)} = \tfrac{1}{2} + \tfrac{1}{2}\, p(z) + \frac{zp'(z)}{p(z) + (2-\lambda)/\lambda} \quad .$$

Thus, defining $q(z) \in P(\lambda-1, -1)$ as in the proof of Theorem 1.4.1 we find

$$1 + \frac{zh''(z)}{h'(z)} = \frac{\lambda-1}{\lambda} + \frac{1}{\lambda}\left[q(z) + \lambda\frac{zq'(z)}{q(z)}\right] \quad .$$

Now we can apply Theorem 1.2.2 with $\alpha = 1$, $\beta = \lambda$, $A = \lambda-1$, $B = -1$ to get

$$\text{(1.4.3)} \quad \mathrm{Re}\left\{1+\frac{zh''(z)}{h'(z)}\right\} \geq \begin{cases} \dfrac{1+(1-2\lambda)r}{[1+(1-\lambda)r](1+r)} & \text{, for } R_1 \leq R_2 \ , \\[2ex] \dfrac{\lambda-1}{\lambda} + \dfrac{1}{\lambda}\left\{\lambda-2- \dfrac{2}{1-r^2}\,[1+(\lambda-1)r^2] \right. \\[2ex] \left. + \dfrac{2}{1-r^2}\,[3(2-\lambda)(1+(\lambda-1)r^2)(1-r^2)]^{\frac{1}{2}}\right\} & \text{, for } R_2 \leq R_1 \ . \end{cases}$$

For $R_2 \leq R_1$, the equation giving the radius of convexity of $h(z)$ may be derived to be

$$G(r) \equiv (1-2\lambda)^2 r^4 - 2(1-6\lambda+2\lambda^2)r^2 + 1 - 8\lambda + 4\lambda^2 = 0 \ .$$

The smallest zero in $(0, 1)$ of $G(r)$ is ρ_2. For $R_1 \leq R_2$, the right-hand side of (1.4.3) is always positive. Hence the radius of convexity of $h(z)$ is ρ_2 .

The result is sharp for the function

$$h_0(z) = \lambda f(z) + (1-\lambda)\int_0^z \frac{f(\xi)}{\xi}\, d\xi \quad ,$$

where

$$f(z) = \frac{z}{1 - 2z\cos\theta + z^2} \quad ,$$

$\cos\theta$ being given by the equation

$$\frac{1 + (2-\lambda)\rho_2\cos\theta + (1-\lambda)\rho_2^2}{1 + 2\rho_2\cos\theta + \rho_2^2} = \left\{\frac{(2-\lambda)[1+(\lambda-1)\rho_2^2]}{3(1-\rho_2^2)}\right\}^{\frac{1}{2}} \quad .$$

CHAPTER 2

K - FOLD SYMMETRIC REGULAR FUNCTIONS

2.1 Introduction

Let k be a positive integer and

(2.1.1) $$f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \ldots + a_{nk+1}z^{nk+1} + \ldots$$

be regular in the unit disc Δ. These functions, which satisfy the relation

$$f(ze^{i2\pi/k}) = e^{i2\pi/k} f(z) \ , \quad z \in \Delta \ ,$$

are called k-fold symmetric functions. We shall be concerned with k-fold symmetric starlike functions. The class of normalised k-fold symmetric starlike univalent functions is denoted by S_k^*.

The study of k-fold symmetric starlike functions was initiated in the early 1930s with the works of Golusin [25], Robertson [73] and Noshiro [61] , each of whom established the coefficient bounds for these functions. Robertson [73] further proved that if $f(z) \in S_k^*$ then

$$\mathrm{Re}\{\frac{f(z)}{z}\}^{k/2} > \tfrac{1}{2} \quad , \quad z \in \Delta \ .$$

Noshiro [61] investigated in great detail geometric properties of the class S_k^* which include bounds for $|f(z)|$, $|f'(z)|$ among other results.

In this chapter, we examine a general subclass of k-fold symmetric starlike functions, namely, the class

$$S_k^*(A, B) = \{f(z) = z + \sum_{n=1}^{\infty} a_{nk+1}z^{nk+1} \ ; \ zf'(z)/f(z) \in P_k(A, B) \ , \quad z \in \Delta\},$$

where $P_k(A, B)$ consists of functions $p(z)$ in $P(A, B)$ which have the series expansion

$$(2.1.2) \qquad p(z) = 1 + p_k z^k + p_{2k} z^{2k} + \ldots + p_{nk} z^{nk} + \ldots$$

With appropriate choices of A and B, the class $S_k^*(A, B)$ reduces to known subclasses of S_k^* ; for example (see Zawadzki [98] and [99]) ,

$$S_k^*(1-2\alpha,-1) \equiv S_{k,\alpha}^* = \{f(z)=z+\sum_{n=1}^{\infty} a_{nk+1} z^{nk+1} \; ; \; \mathrm{Re}\{\frac{zf'(z)}{f(z)}\}>\alpha \; , \; 0\le\alpha<1 \; , \; z\in\Delta\} \; ,$$

$$S_k^*(\alpha,0) \equiv S_{k,(\alpha)}^* = \{f(z)=z+\sum_{n=1}^{\infty} a_{nk+1} z^{nk+1} \; ; \; |\frac{zf'(z)}{f(z)} -1|<\alpha \; , \; 0<\alpha\le 1, \; z\in\Delta\} \; .$$

We shall be concentrating on the following problems:

(i) distortion, covering, radius of convexity for $S_k^*(A, B)$;

(ii) radius of starlikeness of order β, $0 \le \beta < 1$, of the functions

$$F(z) = \lambda f(z) + (1-\lambda) z f'(z) \quad , \quad z \in \Delta \; ,$$

where $-\infty < \lambda < 1$ and $f(z) \in S_k^*(A, B)$.

The consideration of problem (ii) is motivated by recent investigations of Livingston [42] who first established and solved the problem for the case $\lambda = \frac{1}{2}$, $f(z) \in S^*$, Bernardi [10] who extended Livingston's result to the case $\lambda = c/(1+c)$, $c = 1, 2, 3, \ldots$, Goel and Singh [24] who generalised Bernardi's result to the case $c > -1$ and $f(z) \in S^* (\frac{\alpha(1-\beta)+(1-\alpha)\beta}{\alpha} \; , \; \frac{1-\alpha}{\alpha})$, $0 \le \beta < 1$, $\alpha > \frac{1}{2}$.

From the representation

$$\frac{zf'(z)}{f(z)} = p(z) \quad , \; p(z) \in P_k(A, B) \; , \; z \in \Delta$$

for $f(z) \in S_k^*(A, B)$, the problems mentioned above may be reduced to certain special cases of the extremal problem

$$(2.1.3) \qquad \min_{|z|=r<1} \mathrm{Re}\{\alpha p(z) + \beta zp'(z)/p(z)\} \ , \ \alpha \geq 0 \ , \ \beta \geq 0$$

over $P_k(A, B)$. The complete solution to (2.1.3) given in the next section may also be viewed as a generalisation of the extremal problem (1.1.3) considered in Chapter 1.

A limiting case arising from the analysis of problem (ii), namely, the extremal problem

$$(2.1.4) \qquad \min_{|z|=r<1} \mathrm{Re}\{\frac{zp'(z)}{p(z)-1}\}$$

over $P_k(A, B)$, is also dealt with in this chapter. As an application of this problem, we shall prove that if $f(z) \in S^*$ and $f''(0) \neq 0$, then the functions $[f(z) - (2/z)\int_0^z f(\xi)d\xi]^{\frac{1}{2}}$ and $[f(z) - \int_0^z f(\xi)d\xi/\xi]^{\frac{1}{2}}$ belong to $S^*_{\frac{1}{4}}$ and $S^*_{\frac{1}{2}}$ respectively.

We shall extend Dieudonné's lemma to a form which is suitable for the study of k-fold symmetric functions and our solutions to the problems (2.1.3), (2.1.4) are based on this extended lemma.

2.2 The functional $\mathrm{Re}\{\alpha p(z) + \beta zp'(z)/p(z)\}$, $\alpha \geq 0$, $\beta \geq 0$, over $P_k(A, B)$

Let B_k denote the class of regular functions of the form

$$w(z) = b_k z^k + b_{2k} z^{2k} + \ldots + b_{nk} z^{nk} + \ldots$$

such that $|w(z)| < 1$ in Δ. In view of the general Schwarz's lemma, we have $|w(z)| \leq |z|^k$; therefore, we may write

$$w(z) = z^k \psi(z) \quad , \quad z \in \Delta \quad ,$$

where $\psi(z)$ is regular and $|\psi(z)| \leq 1$ in Δ. An application of Carathéodory's inequality (1.2.5) now yields

$$(2.2.1) \qquad |zw'(z)-kw(z)| \leq \frac{|z|^{2k} - |w(z)|^2}{|z|^{k-1}(1-|z|^2)} \; , \quad w(z) \in \mathcal{B}_k \; , \; z \in \Delta \; .$$

Equality in (2.2.1) occurs for functions of the form $z^k(z-c)/(1-cz)$, $|c| \leq 1$.

We recall that

$$P_k(A, B) = \{p(z)=1+\sum_{n=1}^{\infty} p_{nk}z^{nk} \in P(A, B) \; , \; k = 1, 2, 3, \ldots, z \in \Delta\} \; .$$

Thus, for every $p(z) \in P_k(A, B)$, we have

$$(2.2.2) \qquad p(z) = H(w(z)) \quad , \; z \in \Delta \quad ,$$

for some $w(z) \in \mathcal{B}_k$, where $H(z) = (1+Az)/(1+Bz)$. Consequently, an application of the Subordination Principle yields that the image of $|z| \leq r$ under every $p(z) \in P_k(A, B)$ is contained in the disc

$$(2.2.3) \qquad |p(z) - a_k| \leq d_k \quad ,$$

where

$$(2.2.4) \qquad a_k = \frac{1-ABr^{2k}}{1-B^2r^{2k}} \quad , \quad d_k = \frac{(A-B)r^k}{1-B^2r^{2k}} \; .$$

It follows immediately from (2.2.3) and (2.2.4) that if $p(z) \in P_k(A, B)$, then on $|z| = r < 1$,

$$(2.2.5) \qquad \frac{1-Ar^k}{1-Br^k} \le \operatorname{Re}\{p(z)\} \le |p(z)| \le \frac{1+Ar^k}{1+Br^k} \; .$$

The inequalities are sharp for $p(z) = (1+Az^k)/(1+Bz^k)$.

We are now ready to prove our main theorem.

2.2.1 <u>Theorem</u>. *If* $p(z) \in P_k(A, B)$, $\alpha \ge 0$, $\beta \ge 0$, *then on* $|z| = r < 1$

$$\operatorname{Re}\{\alpha p(z)+\beta\frac{zp'(z)}{p(z)}\} \ge \begin{cases} \dfrac{\alpha-[\beta k(A-B)+2\alpha A]r^k+\alpha A^2r^{2k}}{(1-Ar^k)(1-Br^k)} \; , & R_{k1} \le R_{k2} \; , \\[2ex] \beta k\,\dfrac{A+B}{A-B} + 2\,\dfrac{(MN)^{\frac{1}{2}}-\beta(1-ABr^{2k})}{(A-B)r^{k-1}(1-r^2)} \; , & R_{k2} \le R_{k1} \; , \end{cases}$$

where $R_{k1} = (M/N)^{\frac{1}{2}}$, $R_{k2} = (1-Ar^k)/(1-Br^k)$, $M = \beta(1-kAr^{k-1}+kAr^{k+1}-A^2r^{2k})$, $N = \beta + [\alpha(A-B)-\beta kB]r^{k-1}-[\alpha(A-B)-\beta kB]r^{k+1} - \beta B^2r^{2k}$. *The result is sharp.*

<u>Proof</u>. Following the same argument as in the proof of Theorem 1.2.2, we derive from the representation formula (2.2.2) and inequality (2.2.1) that, on $|z| = r$,

$$(2.2.6)\quad \mathrm{Re}\{\alpha p(z) + \beta\frac{zp'(z)}{p(z)}\} \geq \beta k\frac{A+B}{A-B} + \frac{1}{A-B}\,\mathrm{Re}\{[\alpha(A-B)-\beta kB]p(z) - \frac{\beta kA}{p(z)}\}$$

$$- \beta\,\frac{r^{2k}|A-Bp(z)|^2-|p(z)-1|^2}{(A-B)r^{k-1}(1-r^2)|p(z)|}\ .$$

Put $p(z) = a_k + u + iv$, $|p(z)| = R$ and denote the right-hand side of (2.2.6) by $S(u, v)$, then

$$S(u, v) = \beta k\frac{A+B}{A-B} + \frac{1}{A-B}\,\{[\alpha(A-B)-\beta kB](a_k+u) - \frac{\beta kA(a_k+u)}{R^2}$$

$$- \beta\frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)}\ \cdot\ \frac{d_k^2-u^2-v^2}{R}\}\ .$$

Now,

$$(2.2.7)\qquad \frac{\partial S}{\partial v} = \frac{\beta}{A-B}\cdot\frac{v}{R^4}\,T(u, v)\quad ,$$

where

$$T(u, v) = 2kA(a_k+u) + \frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)}\,[2R^3 + (d_k^2-u^2-v^2)R]$$

$$\geq 2(a_k+u)[kA+ \frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)}(a_k - d_k)^2]$$

$$(2.2.8)\qquad = 2(a_k+u)[kA+ \frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)}\cdot(\frac{1-Ar^k}{1-Br^k})^2]\ .$$

We want to show now that

$$(2.2.9)\qquad \frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)} \geq k\ .$$

In fact,

$$\frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)} \geq \frac{1 - r^{2k}}{r^{k-1}(1-r^2)} \geq k$$

if and only if $1 - r^{2k} \geq kr^{k-1}(1 - r^2)$, that is, if and only if

$$F(k, r) \equiv 1 + r^2 + r^4 + \ldots + r^{2(k-1)} - kr^{k-1} \geq 0 .$$

For k even, $F(k, r) = (1-r^{k-1})^2 + r^2(1-r^{k-3})^2 + \ldots + r^{k-2}(1-r)^2 > 0.$

For k odd, $F(k, r) = (1-r^{k-1})^2 + r^2(1-r^{k-3})^2 + \ldots + r^{k-3}(1-r^2)^2 > 0.$

Hence, inequality (2.2.9) always holds. This inequality together with (2.2.8) imply

$$T(u, v) \geq 2k(a_k+u)[A + (\frac{1-Ar^k}{1-Br^k})^2] .$$

Now $A(1-Br^k)^2 + (1-Ar^k)^2 = (1+B)(1-Ar^k)^2 + (A-B)(1-ABr^{2k}) > 0$. Thus $T(u, v) > 0$ and it follows from (2.2.7) that the minimum of $S(u, v)$ on the disc $|p(z) - a_k| \leq d_k$ is attained when $v = 0$, $u \in [-d_k, d_k]$. Setting $v = 0$ in the expression for $S(u, v)$ we get

$$S(u, 0) = \beta k \frac{A+B}{A-B} + \frac{1}{A-B}\{[\alpha(A-B)-\beta kB + \beta\frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)}](a_k+u)$$

$$- \beta[kA - \frac{1-A^2r^{2k}}{r^{k-1}(1-r^2)}]\frac{1}{a_k+u} - 2\beta\frac{1-ABr^{2k}}{r^{k-1}(1-r^2)}\}$$

which yields

$$\frac{dS(u, 0)}{du} = \frac{1}{A-B}\{\alpha(A-B)-\beta kB+\beta\frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)} + \beta[kA - \frac{1-A^2r^{2k}}{r^{k-1}(1-r^2)}]\frac{1}{(a_k+u)^2}\} .$$

We see that the absolute minimum of $S(u, 0)$ occurs at the point $u_0 = (M/N)^{\frac{1}{2}} - a_k$ if u_0 lies inside $[-d_k, d_k]$, its value being

$$S(u_0, 0) = \beta k \frac{A+B}{A-B} + 2\,\frac{(MN)^{\frac{1}{2}} - \beta(1-ABr^{2k})}{(A-B)r^{k-1}(1-r^2)} .$$

We next want to show that $u_0 < d_k$. Indeed, since

$$\alpha(A-B) - \beta kB + \beta\frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)} \geq \beta\left[\frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)} - kB\right]$$

$$\geq \beta k(1-B) \quad , \text{ by (2.2.9)}$$

$$\geq 0$$

and similarly,

$$\frac{1-A^2r^{2k}}{r^{k-1}(1-r^2)} - kA \geq 0 ,$$

we have

$$(a_k+u_0)^2 < \frac{1-kAr^{k-1}+kAr^{k+1}-A^2r^{2k}}{1-kBr^{k-1}+kBr^{k+1}-B^2r^{2k}}$$

$$= \frac{k-Ar^{k-1}}{k-Br^{k-1}}\left(\frac{1-kAr^{k-1}}{k-Ar^{k-1}} + Ar^{k+1}\right)\left(\frac{1-kBr^{k-1}}{k-Br^{k-1}} + Br^{k+1}\right)^{-1} .$$

Since $0 < (k-Ar^{k-1})/(k-Br^{k-1}) < 1$ and the second and third factors are positive, the above inequality reduces to

$$(2.2.10) \quad (a_k+u_0)^2 < \left(\frac{1-kAr^{k-1}}{k-Ar^{k-1}} + Ar^{k+1}\right)\left(\frac{1-kBr^{k-1}}{k-Br^{k-1}} + Br^{k+1}\right)^{-1} .$$

The right-hand side of (2.2.10) is less than or equal to $(1+Ar^{k+1})/(1+Br^{k+1})$ if and only if

$$\frac{1-kAr^{k-1}}{k-Ar^{k-1}} + (1-k)\left(\frac{1+Ar^{k-1}}{k-Ar^{k-1}}\right)Br^{k+1} \le \frac{1-kBr^{k-1}}{k-Br^{k-1}} + (1-k)\left(\frac{1+Br^{k-1}}{k-Br^{k-1}}\right)Ar^{k+1} \quad ,$$

that is, if and only if

$$[(1-k)Br^{k+1}+(1-k)ABr^{2k}+1-kAr^{k-1}](k-Br^{k-1}) \le [(1-k)Ar^{k+1}+(1-k)ABr^{2k} + 1-kBr^{k-1}](k-Ar^{k-1}) \ .$$

This inequality is equivalent to

$$(2.2.11) \qquad (k-1)[1+(A+B)r^{k+1}+ABr^{2k}+k(1-r^2)] \ge 0 \ .$$

Put $G(A, B, r) = 1 + (A+B)r^{k+1} + ABr^{2k}$. Then

$$\frac{\partial G}{\partial B} = r^{k+1}(1+Ar^{k-1}) > 0 \ .$$

Thus,

$$G(A, B, r) \ge G(A, -1, r) = 1-r^{k+1}+Ar^{k+1}(1-r^{k-1})$$
$$\ge (1-r^{k-1})(1+Ar^{k+1}) > 0 \ .$$

This implies that condition (2.2.11) is always satisfied. Consequently, in view of (2.2.10) and these intermediate steps, we have

$$(a_k+u_0)^2 < \frac{1+Ar^{k+1}}{1+Br^{k+1}} \quad .$$

Furthermore, it is clear that

$$\frac{1+Ar^{k+1}}{1+Br^{k+1}} < \frac{1+Ar^k}{1+Br^k} < \left(\frac{1+Ar^k}{1+Br^k}\right)^2 = (a_k+d_k)^2 \ .$$

Hence, $u_0 < d_k$. However, u_0 is not always greater than $-d_k$. For the case $u_0 \le -d_k$, that is, if $R_{k1} \le R_{k2}$, the absolute minimum of $S(u, 0)$ occurs at the end-point $u = -d_k$, the value of which is

$$S(-d_k, 0) = \frac{\alpha-[\beta k(A-B)+2\alpha A]r^k+\alpha A^2r^{2k}}{(1-Ar^k)(1-Br^k)} \ .$$

To see that the result is sharp, we consider the functions

$$p(z) = \frac{1+Az^k}{1+Bz^k} \ , \quad \text{for} \quad R_{k1} \le R_{k2} \ ,$$

$$p(z) = \frac{1+Aw_k(z)}{1+Bw_k(z)} \ , \quad \text{for} \quad R_{k2} \le R_{k1} \ ,$$

where $w_k(z) = z^k(z-c_k)/(1-c_kz)$, with c_k such that $\mathrm{Re}\{[1+Aw_k(z)]/[1+Bw_k(z)]\} = R_{k1}$ at $z = re^{i\pi/k}$.

2.3 Some geometric properties of the class $S_k^*(A, B)$

In this section we derive the sharp bounds for $|f(z)|$, $|f'(z)|$ in the family $S_k^*(A, B)$ and the radius of convexity for $S_k^*(A, B)$. Letting $r \to 1$ in the lower bound for $|f(z)|$ we obtain the disc which is covered by the image of the unit disc under every $f(z)$ in $S_k^*(A, B)$.

2.3.1 Theorem. *Let* $f(z) \in S_k^*(A, B)$, *then on* $|z| = r < 1$,

$$\text{(i)}\quad r(1-Br^k)^{(A-B)/kB} \le |f(z)| \le r(1+Br^k)^{(A-B)/kB}\ , \quad \textit{if}\ B \ne 0\ ,$$

$$r\exp(-\frac{Ar^k}{k}) \le |f(z)| \le r\exp(\frac{Ar^k}{k})\ , \quad \textit{if}\ B = 0\ ;$$

$$\text{(ii)}\quad (1-Ar^k)(1-Br^k)^{[A-(1+k)B]/B} \le |f'(z)| \le (1+Ar^k)(1+Br^k)^{[A-(1+k)B]/B}\ ,$$

$$\textit{if}\ B \ne 0\ ,$$

$$(1-Ar^k)\exp(-\frac{Ar^k}{k}) \le |f'(z)| \le (1+Ar^k)\exp(\frac{Ar^k}{k})\ , \quad \textit{if}\ B = 0\ .$$

<u>Proof</u>. From the structural formula (0.2.10) for $S_k^*(A, B)$ we get

$$\frac{f(z)}{z} = \exp\int_0^z \frac{p(\xi)-1}{\xi}\,d\xi\ , \quad p(z) \in P_k(A, B)\ .$$

Therefore,

$$\left|\frac{f(z)}{z}\right| = \exp[\mathrm{Re}\{\int_0^z \frac{p(\xi)-1}{\xi}\,d\xi\}]\ .$$

Substituting ξ by zt in the integral we have

$$\left|\frac{f(z)}{z}\right| = \exp\int_0^1 \mathrm{Re}\{\frac{p(zt)-1}{t}\}dt\ .$$

It follows from (2.2.5) that, on $|zt| = rt$,

$$\mathrm{Re}\{\frac{p(zt)-1}{t}\} \le \frac{(A-B)r^kt^{k-1}}{1+Br^kt^k}\ .$$

Hence, for $B \ne 0$,

$$\left|\frac{f(z)}{z}\right| \le \exp\int_0^1 \frac{(A-B)r^k t^{k-1}}{1+Br^k t^k}\, dt = (1+Br^k)^{(A-B)/kB} \quad .$$

The lower bound may be obtained similarly. The case $B = 0$ is trivial. To prove (ii), we note that

$$|f'(z)| = \left|\frac{f(z)}{z}\right| \, |p(z)| \quad , \quad p(z) \in P_k(A, B) \ .$$

Hence, applying the above results and (2.2.5), the assertions follow.

All the bounds are sharp for

$$f(z) = z(1+Bz^k)^{(A-B)/kB} \ , \text{ if } B \neq 0 \ ,$$

$$f(z) = z \exp\left(\frac{Az^k}{k}\right) \quad , \quad \text{if } B = 0 \ .$$

The corollary of Theorem 1 of Zawadzki [98] corresponds to the special case $A = 1 - 2\alpha$, $B = -1$.

Letting $r \to 1$ in the lower bound for $|f(z)|$ we obtain the covering theorem for $S_k^*(A, B)$.

2.3.2 Corollary. *The image of the unit disc under a function* $f(z) \in S_k^*(A, B)$ *contains the disc of centre* 0 *and radius* $(1-B)^{(A-B)/kB}$ *if* $B \neq 0$, $\exp(-A/k)$ *if* $B = 0$.

As defined in 0.2.3, the radius of convexity of $S_k^*(A, B)$ is given by the smallest root in $(0, 1]$ of the equation $\Omega(r) = 0$, where

$$\Omega(r) = \min\{\mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\} \quad ; \quad |z| = r < 1 \ , \ f(z) \in S_k^*(A, B)\}$$

$$= \min\{\mathrm{Re}\{p(z)+\frac{zp'(z)}{p(z)}\}; \ |z| = r < 1 \ , \ p(z) \in P_k(A, B)\} \ .$$

An application of Theorem 2.2.1 with $\alpha = 1$, $\beta = 1$ gives $\Omega(r)$ and solving $\Omega(r) = 0$ we obtain

2.3.3 Corollary. *The radius of convexity of* $S_k^*(A, B)$ *is given by the smallest root in* $(0, 1]$ *of*

(i) $1 - [(2+k)A-kB]r^k + A^2r^{2k} = 0$, *if* $R_{k1} \le R_{k2}$,

(ii) $-4 + 4r^2 + k[k(A-B)+4A]r^{k-1} - 2[(k^2-2)(A-B)+4kA]r^{k+1}$

$+ k[k(A-B)+4A]r^{k+3} + 4A^2r^{2k} - 4A^2r^{2k+2} = 0$, *if* $R_{k2} \le R_{k1}$,

where R_{k1}, R_{k2} *are as given in Theorem* 2.2.1 .

Putting $k = 1$, $A = 1-2\alpha$, $B = -1$ in Corollary 2.3.3 we find that the radius of convexity of the class S_α^* of starlike functions of order α is

$$r_c = \begin{cases} \{2-3\alpha+[(1-\alpha)(3-5\alpha)]^{\frac{1}{2}}\}^{-1} \ , & 0 \le \alpha \le \alpha_0 \ , \\ \left[\dfrac{5\alpha - 1}{1-\alpha+4\alpha^2+4\alpha(2-3\alpha+\alpha^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}} , & \alpha_0 \le \alpha < 1 \ , \end{cases}$$

where α_0 is the smallest positive root of the equation

$$20\alpha^4 - 52\alpha^3 + 15\alpha^2 + 12\alpha - 4 = 0 .$$

This result was obtained previously by Zmorovič [100] and Singh and Goel [85] .

2.4 On Livingston's problem

Fairly recently, Libera [39] showed that if $f(z) \in S^*$ then the function

$$(2.4.1) \qquad g(z) = \frac{2}{z}\int_0^z f(\xi)d\xi$$

is also in S^*. Livingston [42]studied the converse problem, namely, if $g(z) \in S^*$, what is the radius of starlikeness of the function

$$(2.4.2) \qquad f(z) = \tfrac{1}{2}[g(z) + zg'(z)] \quad ?$$

Livingston showed that $f(z)$ is starlike in $|z| < \frac{1}{2}$. This result has been refined and generalised in different ways by many authors. Padmanabhan [63] proved that if $g(z) \in S_\alpha^*$, $0 \le \alpha \le \frac{1}{2}$, then $f(z)$, as defined by (2.4.2), is starlike of the same order in $|z| < [\alpha-2+(\alpha^2+4)^{\frac{1}{2}}]/2\alpha$. Libera and Livingston [40] extended Padmanabhan's result to include the range $\frac{1}{2} < \alpha < 1$. These authors obtained the radius of the disc in which $f(z)$ is starlike of order β, $f(z)$ being as given by (2.4.2) with $g(z) \in S_\alpha^*$, $0 \le \alpha < 1$ and $\beta \ge \alpha$. The complementary case $0 \le \beta < \alpha$ was ~~proved~~ studied by Al-Amiri [2] and Bajpai and Singh [6] .

In another direction, Bernardi [10] found the radius of starlikeness of the functions $f(z)$ defined by

$$(2.4.3) \qquad f(z) = \frac{1}{1+c}\,[c\,g(z)+zg'(z)]\ ,$$

where $c = 1, 2, 3, \ldots$ and $g(z) \in S^*$. Goel and Singh [24] extended and generalised Bernardi's result to the case in which c is any real number such that $c + \beta > 0$ and $g(z)$ belongs to a more restricted family characterised by the condition

$$\left|\left(\frac{zg'(z)}{g(z)} - \beta\right)/(1-\beta) - \alpha\right| < \alpha\ ,\ 0 \le \beta < 1\ ,\ \alpha > \tfrac{1}{2}\ .$$

We note that this class is a special case of $S^*(A, B)$ with $A = [\alpha(1-\beta)+(1-\alpha)\beta]/\alpha$, $B = (1-\alpha)/\alpha$. We further remark that for $1 + c > 0$ and putting $\lambda = c/(1+c)$, equation (2.4.3) is equivalent to

$$(2.4.4) \qquad f(z) = \lambda g(z) + (1-\lambda)zg'(z)\ ,\ -\infty < \lambda < 1\ .$$

The restriction $c + \beta > 0$ in Goel and Singh's analysis corresponds to $\beta/(\beta-1) < \lambda < 1$.

In the following, as another direct application of Theorem 2.2.1 , we determine the sharp radius of the disc in which every $f(z)$ as given by (2.4.4) with $g(z) \in S_k^*(A, B)$ is starlike of order γ , $0 \le \gamma < 1$. All the above-mentioned results are special cases of this with $k = 1$ and appropriately chosen values of A, B, γ .

2.4.1 <u>Theorem</u>. *Let* $f(z) = \lambda g(z) + (1-\lambda)zg'(z)$, *where* $(A-1)/(A-B) \le \lambda < 1$

and $g(z) \in S_k^*(A, B)$. *Let* r_{k1} *be the smallest root in* $(0, 1]$ *of the equation*

$$(1-\lambda)(1-\gamma)+[(\lambda+\gamma(1-\lambda))(B+C)-k(1-\lambda)(C-B)-2C]r^k+[C^2-(\lambda+\gamma(1-\lambda))BC]r^{2k} = 0$$

and r_{k2} *the smallest root in* $(0, 1]$ *of the equation*

$$4(1-\lambda)[D-E+(1-\lambda)kC]-4(1-\lambda)[D-E+(1-\lambda)kC]r^2+[D^2+4(1-\lambda)kCE]r^{k-1}$$
$$+[4(1-\lambda)^2(C-B)^2-2D^2-8(1-\lambda)kCE]r^{k+1}+[D^2+4(1-\lambda)kCE]r^{k+3}$$
$$+4(1-\lambda)[C^2E-CBD-(1-\lambda)kCB^2]r^{2k}-4(1-\lambda)[C^2E-CBD-(1-\lambda)kCB^2]r^{2k+2} = 0 ,$$

where $C = (1-\lambda)A + \lambda B$, $D = [\lambda+\gamma(1-\lambda)](C-B)-k(1-\lambda)(C+B)$, $E=C-B-k(1-\lambda)B$. *Then* $f(z)$ *is starlike of order* γ, $0 \le \gamma < 1$, *in*

$$|z| < \begin{cases} r_{k1} , \textit{ for } R_{k1} \le R_{k2} , \\ r_{k2} , \textit{ for } R_{k2} \le R_{k1} , \end{cases}$$

R_{k1}, R_{k2} *being as given in Theorem* 2.2.1 *with* A *replaced by* C , $\alpha = 1$, $\beta = 1-\lambda$.

<u>Proof</u>. Since $g(z) \in S_k^*(A, B)$, we may write

$$\frac{zg'(z)}{g(z)} = p(z) \quad , p(z) \in P_k(A, B) .$$

Then, from the definition of $f(z)$, we have

$$(2.4.5) \qquad \frac{zf'(z)}{f(z)} = p(z) + \frac{zp'(z)}{p(z)+\mu}\,, \quad \mu = \frac{\lambda}{1-\lambda}\,, \ -1 < \mu < \infty\,.$$

Put $q(z) = [p(z)+\mu]/(1+\mu)$. Then, in terms of functions of B_k,

$$q(z) = \frac{1+[(1-\lambda)A+\lambda B]w(z)}{1+Bw(z)}\,, \quad w(z) \in B_k\,.$$

Hence $q(z) \in P_k(C, B)$ and

$$(2.4.6) \qquad \frac{zq'(z)}{q(z)} = \frac{zp'(z)}{p(z)+\mu} = \frac{(1-\lambda)(A-B)zw'(z)}{[1+Bw(z)][1+((1-\lambda)A+\lambda B)w(z)]}\,.$$

It is clear from (2.4.6) that the function $zp'(z)/[p(z)+\mu]$ is not regular in Δ if $(1-\lambda)A + \lambda B > 1$, that is, if $\lambda < (A-1)/(A-B)$. Hence we confine λ to the range $(A-1)/(A-B) \le \lambda < 1$ so that $zp'(z)/[p(z)+\mu]$ is regular in the entire unit disc. Equation (2.4.5) may be rewritten as

$$(2.4.7) \qquad \frac{zf'(z)}{f(z)} = -\frac{\lambda}{1-\lambda} + \frac{1}{1-\lambda}\left[q(z)+(1-\lambda)\frac{zq'(z)}{q(z)}\right],\ q(z) \in P_k(C, B)\,.$$

Now, the radius of starlikeness of order γ of $f(z)$ is determined by the equation

$$\min_{f(z)\in S_k^*(A,\,B)} \ \min_{|z|=r<1} \ \mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \gamma\right\} = 0\,,$$

or equivalently, from (2.4.7),

$$(2.4.8) \qquad \min_{q(z)\in P_k(C,\,B)} \ \min_{|z|=r<1} \ \mathrm{Re}\left\{-\gamma-\frac{\lambda}{1-\lambda} + \frac{1}{1-\lambda}\left[q(z)+(1-\lambda)\frac{zq'(z)}{q(z)}\right]\right\} = 0\,.$$

Hence an application of Theorem 2.2.1 with A replaced by C, $\alpha = 1$, $\beta = 1-\lambda$ to (2.4.8) will yield the equations giving the starlikeness of f(z). The sharpness of the result follows from that of Theorem 2.2.1 .

2.4.2 <u>Remark</u>. Let us look at some special cases of Theorem 2.4.1 . We first consider the case $k = 1$, $\gamma = 0$, $g(z) \in S^*$. Then $A = 1$, $B = -1$, $C = 1-2\lambda$. Thus the equation giving r_{k1} is reduced to

$$1 - 4(1-\lambda)r + (1-2\lambda)r^2 = 0 . \tag{2.4.9}$$

For $R_{k2} \le R_{k1}$, we find

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} \ge -\frac{\lambda}{1-\lambda} + \frac{1}{1-\lambda}\{-\lambda + \frac{2[2\lambda(1-r^2)(1+(1-2\lambda)r^2)]^{\frac{1}{2}} - [1+(1-2\lambda)r^2]}{1-r^2}\}$$

$$= -\{(\frac{2\lambda}{1-\lambda})^{\frac{1}{2}} - [\frac{1+(1-2\lambda)r^2}{(1-\lambda)(1-r^2)}]^{\frac{1}{2}}\}^2$$

which is always negative. Hence the radius of starlikeness of f(z) in this special case is given by equation (2.4.9), which is

$$r_{11} = \frac{2(1-\lambda)-(3-6\lambda+4\lambda^2)^{\frac{1}{2}}}{1-2\lambda} ,$$

or, replacing λ by $c/(1+c)$, $r_{11} = [2-(3+c^2)^{\frac{1}{2}}]/(1-c)$. This is the result obtained previously by Bernardi [10] using another method.

We next consider the case $k = 1$, $\lambda = \frac{1}{2}$, $\gamma = 0$ and $g(z) \in S_\alpha^*$. Then $A = 1-2\alpha$, $B = -1$, $C = -\alpha$ and Theorem 2.4.1 yields the radius of starlikeness of f(z) in this special case to be

$$r_* = \begin{cases} \{[(1-\alpha)(1-2\alpha)]^{\frac{1}{2}} + 1-2\alpha\}^{-1} , & 0 \le \alpha \le \alpha_1 , \\ \{\dfrac{2\alpha}{[\alpha(2-\alpha)(1-\alpha^2)]^{\frac{1}{2}} + \alpha(1+\alpha)}\}^{\frac{1}{2}} , & \alpha_1 \le \alpha < 1 , \end{cases}$$

where α_1 is the smallest positive root of the equation

$$4\alpha^3 - 4\alpha^2 - 10\alpha + 1 = 0 .$$

This result was derived earlier by Singh and Goel [85] .

2.5 The functional $\mathrm{Re}\{zp'(z)/[p(z)-1]\}$ over $P_k(A, B)$

In Theorem 2.4.1 of the previous section, we have encountered implicitly the functional

$$\text{(2.5.1)} \qquad \mathrm{Re}\{\frac{zp'(z)}{p(z)+\mu}\} , \quad -1 < \mu < \infty \quad ,$$

over $P_k(A, B)$. Since the function $zp'(z)/[p(z)+\mu]$ is not regular in Δ for $-1 < \mu < 0$, we have to impose certain conditions on μ so that the function becomes regular in the entire unit disc. However, for the limiting case $\mu = -1$, this restriction is no longer necessary; for now, from the normalisation of $p(z)$, we have a removable singularity at $z = 0$.

It is the purpose of this section to obtain the lower bound for the functional (2.5.1) with $\mu = -1$ on $|z| = r < 1$. We shall see that this lower bound is always positive. Hence the result that every $p(z) \in P_k(A, B)$ is starlike with respect to the point 1 in Δ is readily established. The bound obtained is further used to study the starlikeness of an operator over the class S^*, namely,

$$F(z) = [f(z)-(2/z)\int_0^z f(\xi)d\xi]^{\frac{1}{2}} .$$

2.5.1 <u>Theorem</u>. *If* $p(z) \in P_k(A, B)$, *then on* $|z| = r < 1$

$$\mathrm{Re}\{\frac{zp'(z)}{p(z)-1}\} \geq \begin{cases} k/(1-Br^k) \ , \ B \leq 0 \ , \\ k/(1+Br^k) \ , \ B \geq 0 \ . \end{cases}$$

<u>Proof</u>. For $p(z) \in P_k(A, B)$, we derive from the representation formula (2.2.2) that

$$(2.5.2) \qquad \frac{zp'(z)}{p(z)-1} = \frac{zw'(z)}{w(z)[1+Bw(z)]} \ , \quad w(z) \in B_k \ .$$

An application of (2.2.1) now yields on $|z| = r$

$$(2.5.3) \quad \mathrm{Re}\{\frac{zw'(z)}{w(z)[1+Bw(z)]}\} \geq \mathrm{Re}\{\frac{k}{1+Bw(z)}\} - \frac{r^{2k} - |w(z)|^2}{r^{k-1}(1-r^2)|w(z)||1+Bw(z)|} \ .$$

Put $w_1(z) = w(z)/[1+Bw(z)]$; then (2.5.2) and (2.5.3) imply

$$(2.5.4) \quad \mathrm{Re}\{\frac{zp'(z)}{p(z)-1}\} \geq k - kB\,\mathrm{Re}\{w_1(z)\} - \frac{r^{2k}|1-Bw_1(z)|^2-|w_1(z)|^2}{r^{k-1}(1-r^2)|w_1(z)|} \ .$$

The image of $|z| \leq r$ under the transformation $w_1(z)$ is contained in the disc $|w_1(z) - \alpha| \leq \delta$, where

$$\alpha = -\frac{Br^{2k}}{1 - B^2r^{2k}} \ , \quad \delta = \frac{r^k}{1 - B^2r^{2k}} \ .$$

Thus putting $w_1(z) = \alpha + u + iv$, $|w_1(z)| = R$ and denoting the right-hand side of (2.5.4) by $S(u, v)$ we obtain

(2.5.5) $$S(u, v) = k - kB(\alpha+u) - \frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)} \cdot \frac{\delta^2-u^2-v^2}{R} ,$$

which yields

$$\frac{\partial S}{\partial v} = \frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)} \cdot \frac{v}{R^3} T(u, v) ,$$

where $T(u, v) = 2R^2 + \delta^2 - u^2 - v^2 > 0$. Hence the minimum of $S(u, v)$ on the disc $|w_1(z) - \alpha| \leq \delta$ is attained when $v = 0$ and $u \in [-\delta, \delta]$. Setting $v = 0$ in (2.5.5) we get

$$S(u, 0) = \begin{cases} k - kB(\alpha+u) + \dfrac{1-B^2r^{2k}}{r^{k-1}(1-r^2)} \cdot \dfrac{u^2 - \delta^2}{\alpha + u} , & \alpha + u > 0 , \\[2ex] k - kB(\alpha+u) + \dfrac{1-B^2r^{2k}}{r^{k-1}(1-r^2)} \cdot \dfrac{\delta^2 - u^2}{\alpha + u} , & \alpha + u < 0 . \end{cases}$$

Replacing u by $(\alpha+u) - \alpha$, $S(u, 0)$ becomes

$$S(u, 0) = \begin{cases} k + \dfrac{2Br^{2k}}{r^{k-1}(1-r^2)} - \dfrac{r^{2k}}{r^{k-1}(1-r^2)} \cdot \dfrac{1}{\alpha+u} + [\dfrac{1-B^2r^{2k}}{r^{k-1}(1-r^2)} - kB](\alpha+u) , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \alpha + u > 0 , \\[2ex] k - \dfrac{2Br^{2k}}{r^{k-1}(1-r^2)} + \dfrac{r^{2k}}{r^{k-1}(1-r^2)} \cdot \dfrac{1}{\alpha+u} - [\dfrac{1-B^2r^{2k}}{r^{k-1}(1-r^2)} + kB](\alpha+u) , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \alpha + u < 0 . \end{cases}$$

As proved previously in Theorem 2.2.1, inequality (2.2.9) implies

(2.5.6) $$\frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)} \pm kB \geq k(1 \pm B) > 0 .$$

Thus, in view of (2.5.6), it follows that $d\,S(u, 0)/du > 0$ for $\alpha + u > 0$; and so, the minimum of $S(u, 0)$ occurs at the point $u = -\delta$, its value being $S(-\delta, 0) = k/(1-Br^k)$. Similarly, $d\,S(u, 0)/du < 0$ for $\alpha + u < 0$; hence the minimum of $S(u, 0)$ in this case is attained at $u = \delta$, its value being $S(\delta, 0) = k/(1+Br^k)$. It is clear that $S(-\delta, 0) < S(\delta, 0)$ for $B \le 0$ and vice versa for $B \ge 0$. Hence the result follows and is easily seen to be sharp for the function $p(z) = (1+Az^k)/(1+Bz^k)$.

In [77], Robertson proved that if $K(z)$ denotes the Koebe function $z/(1-z)^2$, which is univalent and starlike in Δ, then the functions

$$S(z) = \frac{2}{z}\int_0^z K(\xi)d\xi \quad ,$$

$$T(z) = [K(z) - S(z)]^{\frac{1}{2}}$$

are also univalent and starlike in Δ.

The extremal character of the Koebe function $K(z)$ within the class S^* suggests the generalisation of these results to functions of the entire class S^*. In fact, as noted previously, Libera [39] showed that if $f(z) \in S^*$, then $(2/z)\int_0^z f(\xi)d\xi \in S^*$. Here, also as an application of Theorem 2.5.1, we prove

2.5.2 <u>Theorem</u>. *If* $f(z) \in S^*$ *with* $f''(0) \neq 0$, *then the function*

$$F(z) = [f(z) - (2/z)\int_0^z f(\xi)d\xi]^{\frac{1}{2}}$$

is univalent starlike of order $\frac{1}{4}$ *in* Δ .

Proof. The condition $f''(0) \neq 0$ ensures that $F'(z) \neq 0$ in Δ. Put $g(z) = (2/z)\int_0^z f(\xi)d\xi$; then

$$2f(z) = g(z) + zg'(z) .$$

Hence $F(z)$ can be rewritten as

$$2F^2(z) = -g(z) + zg'(z) ,$$

from which we deduce

$$\frac{zF'(z)}{F(z)} = \tfrac{1}{2} . \frac{z^2g''(z)}{g(z)} . [\frac{zg'(z)}{g(z)} - 1]^{-1} .$$

Since $g(z) \in S^*$ as shown by Libera [39] , we have $zg'(z)/g(z) = p(z)$, $z \in \Delta$, for some $p(z) \in P$ and

$$\frac{zF'(z)}{F(z)} = \tfrac{1}{2}[p(z) + \frac{zp'(z)}{p(z)-1}] .$$

Taking into account (2.2.5) and Theorem 2.5.1 with $k = 1$, $A = 1$, $B = -1$, we get on $|z| = r$

$$\mathrm{Re}\{\frac{zF'(z)}{F(z)}\} \geq \tfrac{1}{2} . \frac{2-r}{1+r} > \tfrac{1}{4} \quad , \text{for} \quad 0 < r < 1 .$$

Hence $F(z)$ is univalent starlike of order $\frac{1}{4}$ in Δ.

If the operator $(2/z)\int_0^z f(\xi)d\xi$ is replaced by $\int_0^z f(\xi)d\xi/\xi$ in the expression for $F(z)$, then since $f(z) \in S^*$ implies $\int_0^z f(\xi)d\xi/\xi \in S^*_{\frac{1}{2}}$, the same argument as in Theorem 2.5.2 will yield

$$\frac{zF'(z)}{F(z)} = \tfrac{1}{2}[\tfrac{1}{2} + \tfrac{1}{2}\, p(z) + \frac{zp'(z)}{p(z)-1}] , \quad p(z) \in P .$$

Thus in this case we have, on $|z| = r < 1$,

$$\mathrm{Re}\{\frac{zF'(z)}{F(z)}\} \geq \frac{1}{1+r} > \tfrac{1}{2} .$$

In other words ,

2.5.3 <u>Theorem</u>. *If* $f(z) \in S^*$ *with* $f''(0) \neq 0$, *then the function*

$$F(z) = [f(z) - \int_0^z f(\xi)d\xi/\xi]^{\frac{1}{2}}$$

is univalent starlike of order $\tfrac{1}{2}$ *in* Δ .

CHAPTER 3

REGULAR FUNCTIONS WITH A FIXED COEFFICIENT

3.1 Introduction

Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be in $P(A, B)$ and put $\theta = \arg p_1$. Then $p(e^{-i\theta}z) = 1 + |p_1|z + \ldots \in P(A, B)$. Hence there is no loss of generality in limiting our study to functions in $P(A, B)$ with a non-negative real first coefficient. Also, it is known that $|p_1| \leq A - B$ (see, for example, Libera and Livingston [41]). From these observations, we define another subclass of $P(A, B)$, namely,

$$P_b(A, B) = \{p(z) \in P(A, B) \; ; \; p'(0) = b(A-B) \; , \; 0 \leq b \leq 1\} \; .$$

We shall be concerned with the extremal problem

$$(3.1.1) \qquad \min_{|z|=r<1} \; \mathrm{Re}\{\alpha p(z) + \beta z p'(z)/p(z)\} \; , \; \alpha \geq 0 \; , \; \beta \geq 0$$

over $P_b(A, B)$. This is a refinement of the extremal problem (1.1.3) considered in Chapter 1. As it turns out, the solution for (3.1.1) with $b = 1$ is precisely that for (1.1.3) given in Theorem 1.2.2 .

Our approach to problem (3.1.1) is again based on Dieudonné's lemma which applies here even though the constraint $p'(0) = b(A-B)$ is imposed on $P(A, B)$. This may be seen from the fact that the function $w_b = z(z+b)/(1+bz)$ gives equality in (1.2.4) and the corresponding function $p(z) = [1 + Aw_b(z)]/[1 + Bw_b(z)]$ belongs to $P_b(A, B)$.

For some applications of this extremal problem, we shall consider two subclasses of univalent functions with fixed second coefficient generated from $P_b(A, B)$, namely,

$$S_b^*(A, B) = \{f(z)=z+b(A-B)z^2+\ldots \; ; \; zf'(z)/f(z) \in P_b(A, B) \; , \; z \in \Delta\} \; ,$$

$$P_b'(A, B) = \{f(z)=z+\tfrac{b}{2}(A-B)z^2+\ldots \; ; \; f'(z) \in P_b(A, B) \; , \; z \in \Delta\} \; .$$

We shall investigate how the second coefficient in the series expansion of the functions in these classes affects certain properties such as distortion, covering and convexity of these functions. This type of problems was first studied by Gronwall [28] on univalent and convex functions. Finkelstein [20] obtained distortion theorems for $S_b^*(1, -1)$. These results were generalised to the class $S_b^*(1-2\alpha, -1)$, $0 \le \alpha < 1$, of starlike functions of order α with fixed second coefficient by Tepper [90], who also derived the radius of convexity of $S_b^*(1, -1)$. The radius of convexity of $S_b^*(1-2\alpha, -1)$ was found by McCarty [51]. The latter author also obtained corresponding results for the class $P_b'(1-2\alpha, -1)$ of functions whose derivative has real part greater than α in Δ. Our results for $S_b^*(A, B)$ and $P_b'(A, B)$ will naturally cover all these as special cases.

In the final section of this chapter, we establish the radius of convexity of the class of functions $f(z) = z - 2bz^2 + \ldots$, $0 \le b \le 1$, which satisfy $\mathrm{Re}\{f(z)/z\} > 0$ in Δ. This refines a result due to Reade, Ogawa and Sakaguchi [72].

3.2 The functional $\mathrm{Re}\{\alpha p(z)+\beta zp'(z)/p(z)\}$, $\alpha \ge 0$, $\beta \ge 0$, over $P_b(A, B)$

For $p(z) \in P_b(A, B)$, we may write

(3.2.1) $$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad , \quad z \in \Delta \quad ,$$

for some $w(z) \in \mathcal{B}$ so that

$$w(z) = \frac{1 - p(z)}{Bp(z) - A} = bz + \dots \quad = z\psi(z) \ ,$$

where $\psi(z)$ is regular and $|\psi(z)| \le 1$ in Δ with $\psi(0) = b$. Now, since $0 \le b \le 1$, we have

$$\frac{\psi(z) - b}{1 - b\psi(z)} \prec z \ , \quad z \in \Delta \quad .$$

Hence

$$\psi(z) \prec \frac{z + b}{1 + bz} \quad , \quad z \in \Delta \ ,$$

which yields

(3.2.2) $$\mathrm{Re}\{\psi(z)\} \ge \frac{b-|z|}{1-b|z|} \ , \ |\psi(z)| \le \frac{|z|+b}{1+b|z|} \ , \ |w(z)| \le |z|\frac{|z|+b}{1+b|z|} \ .$$

We next put $D = (r+b)/(1+br)$, $0 < r < 1$, and define

$$H_r(z) = \frac{1 + ADz}{1 + BDz} \quad , \quad z \in \Delta \ ;$$

then it is clear that

(3.2.3) $$p(z) \prec H_r(z) \quad , \quad |z| \le r \ .$$

And so, $p(z)$ maps $|z| \le r$ into the disc

(3.2.4) $$|p(z) - a_b| \le d_b \quad ,$$

where

(3.2.5) $$a_b = \frac{1-ABC^2}{1-B^2C^2} \;,\quad d_b = \frac{(A-B)C}{1-B^2C^2} \;,\quad C = r\frac{r+b}{1+br} \;.$$

It follows immediately from (3.2.4) and (3.2.5) that if $p(z) \in P_b(A, B)$, then on $|z| = r < 1$,

(3.2.6) $$\frac{1-AC}{1-BC} \le \operatorname{Re}\{p(z)\} \le |p(z)| \le \frac{1+AC}{1+BC} \quad .$$

The first inequality is sharp for the function

$$p(z) = \frac{1+b(A-1)z-Az^2}{1+b(B-1)z-Bz^2} \quad \text{at } z = -r$$

while the third inequality is sharp for the function

$$p(z) = \frac{1+b(1+A)z+Az^2}{1+b(1+B)z+Bz^2} \quad \text{at } z = r \;.$$

Also, putting $E(b) = a_b-d_b = (1-AC)/(1-BC)$, $F(b) = a_b+d_b = (1+AC)/(1+BC)$, C being as given by (3.2.5), we have

$$\frac{dC}{db} = \frac{r(1-r^2)}{(1+br)^2} > 0 \;,\; \frac{dE}{db} = -\frac{A-B}{(1-BC)^2}\cdot\frac{dC}{db} < 0 \;,\; \frac{dF}{db} = \frac{A-B}{(1+BC)^2}\cdot\frac{dC}{db} > 0 \;.$$

Thus for a fixed r in $(0, 1)$,

(3.2.7) $$a_b - d_b \ge a_1 - d_1 \quad,\quad a_b + d_b \ge a_0 + d_0 \quad .$$

We now prove

3.2.1 Theorem. *If* $p(z) \in P_b(A, B)$, $\alpha \geq 0$, $\beta \geq 0$, *then on* $|z| = r < 1$,

$$\mathrm{Re}\{\alpha p(z)+\beta\frac{zp'(z)}{p(z)}\} \geq \begin{cases} \beta\frac{A+B}{A-B} + \frac{1}{(A-B)(1-r^2)}[L_1\cdot\frac{1-BC}{1-AC} + K_1\cdot\frac{1-AC}{1-BC} - 2\beta(1-ABr^2)] , \\ \qquad\qquad R_1 \leq R_2' , \\ \beta\frac{A+B}{A-B} + \frac{2}{(A-B)(1-r^2)}[(L_1K_1)^{\frac{1}{2}} - \beta(1-ABr^2)] , \quad R_2' \leq R_1 , \end{cases}$$

where $R_1 = (L_1/K_1)^{\frac{1}{2}}$, $R_2' = (1-AC)/(1-BC)$, $L_1 = \beta(1-A)(1+Ar^2)$, $K_1 = \alpha(A-B)(1-r^2) + \beta(1-B)(1+Br^2)$, $C = r(r+b)/(1+br)$. *The result is sharp.*

Proof. With the same argument as in the proof of Theorem 1.2.2, we derive from the representation formula (3.2.1) and Dieudonné's lemma that, on $|z| = r$,

$$\mathrm{Re}\{\alpha p(z)+\beta\frac{zp'(z)}{p(z)}\} \geq \beta\frac{A+B}{A-B} + \frac{1}{A-B}\,\mathrm{Re}\{[\alpha(A-B)-\beta B]p(z) - \frac{\beta A}{p(z)}\} \tag{3.2.8}$$

$$- \beta\frac{r^2|Bp(z)-A|^2 - |1-p(z)|^2}{(A-B)(1-r^2)|p(z)|} .$$

In view of (3.2.4), we put $p(z) = a_b + u + iv$, $|p(z)| = R$, then

$$r^2|Bp(z)-A|^2 - |1-p(z)|^2 = -(1-B^2r^2)R^2 + 2(1-ABr^2)(a_b+u) - (1-A^2r^2)$$

$$= -(1-B^2r^2)R^2+2a_1(1-B^2r^2)(a_b+u)-(1-B^2r^2)(a_1^2-d_1^2).$$

Thus, denoting the right-hand side of (3.2.8) by $S(u, v)$, we get

$$S(u,\ v) = \beta\frac{A+B}{A-B} + \frac{1}{A-B}\{[\alpha(A-B)-\beta B](a_b+u) - \frac{\beta A(a_b+u)}{R^2}$$

$$+ \beta\ \frac{1-B^2r^2}{1-r^2}\ [R-2a_1.\frac{a_b+u}{R} + \frac{a_1^2-d_1^2}{R}]\}$$

$$= \beta\frac{A+B}{A-B} + \frac{1}{A-B}\{[\alpha(A-B)-\beta B-\frac{\beta A}{R^2}](a_b+u)+\beta\frac{1-B^2r^2}{1-r^2}\cdot\frac{1}{R}[(a_b+u-a_1)^2$$

$$+ v^2-d_1^2]\}\ .$$

This gives

$$\frac{\partial S}{\partial v} = \frac{\beta}{A-B}\cdot\frac{v}{R^4}\,T(u,\ v) \tag{3.2.9}$$

where

$$T(u,\ v) = 2A(a_b+u)+\frac{1-B^2r^2}{1-r^2}\{R^3-R[a_1^2-2(a_b+u)a_1-d_1^2]\}$$

$$= 2(a_b+u)(A+\frac{1-B^2r^2}{1-r^2}\cdot a_1R) + \frac{1-B^2r^2}{1-r^2}\ [R^3-R(a_1^2 - d_1^2)]\ .$$

Since $R \geq a_b - d_b \geq a_1 - d_1$ as seen from (3.2.7), it follows that

$$A + \frac{1-B^2r^2}{1-r^2}\cdot a_1R \geq A+(a_1-d_1)^2 = \frac{(1+B)(1-Ar)^2+(A-B)(1-ABr^2)}{(1-Br)^2} > 0. \tag{3.2.10}$$

Consequently,

$$T(u,\ v) \geq 2(a_1-d_1)(A+\frac{1-B^2r^2}{1-r^2}\cdot a_1R)+\frac{1-B^2r^2}{1-r^2}\ [R^3-R(a_1^2-d_1^2)]\ .$$

Denote the right-hand side by G(R), then

$$\frac{dG}{dR} = \frac{1-B^2r^2}{1-r^2}[(a_1-d_1)^2 + 3R^2] > 0 .$$

Thus, by (3.2.10)

$$G(R) \geq G(a_1-d_1) = 2(a_1-d_1)[A+ \frac{1-B^2r^2}{1-r^2}(a_1-d_1)^2] > 0 .$$

Hence $T(u, v) > 0$, and in view of (3.2.9), we see that the minimum of $S(u, v)$ on the disc $|p(z) - a_b| \leq d_b$ is attained when $v = 0$ and $u \in [-d_b, d_b]$. Setting $v = 0$, we get

$$S(u, 0) = \beta\frac{A+B}{A-B} + \frac{1}{A-B}\{\beta\frac{(1-A)(1+Ar^2)}{1-r^2} \cdot \frac{1}{a_b+u} + \frac{\alpha(A-B)(1-r^2)+\beta(1-B)(1+Br^2)}{1-r^2}.(a_b+u)$$

$$- 2\beta\frac{1-ABr^2}{1-r^2}\}$$

which yields

$$\frac{dS(u, 0)}{du} = \frac{1}{(A-B)(1-r^2)}[- \frac{L_1}{(a_b+u)^2} + K_1] .$$

It is clear that the absolute minimum of $S(u, 0)$ occurs at the point $u_0 = (L_1/K_1)^{\frac{1}{2}} - a_b$ if u_0 lies in $[-d_b, d_b]$, its value being

$$S(u_0, 0) = \beta\frac{A+B}{A-B} + \frac{2}{(A-B)(1-r^2)}[(L_1K_1)^{\frac{1}{2}}-\beta(1-ABr^2)] .$$

Now, from (1.2.10) and (3.2.7) we have that

$$(a_b+u_0)^2 < \frac{1+Ar^2}{1+Br^2} = a_0 + d_0 \leq a_b + d_b \leq (a_b + d_b)^2 .$$

Thus $u_0 < d_b$. However, it is not necessary that $u_0 > -d_b$. For the case $u_0 \le -d_b$, that is, if $R_1 \le R_2'$, the absolute minimum of $S(u, 0)$ occurs at the end-point $u = -d_b$, the value of which is

$$S(-d_b, 0) = \beta\frac{A+B}{A-B} + \frac{1}{(A-B)(1-r^2)}[L_1 . \frac{1-BC}{1-AC} + K_1 . \frac{1-AC}{1-BC} - 2\beta(1-ABr^2)] .$$

The result is sharp for the function

$$p(z) = \frac{1+b(A-1)z-Az^2}{1+b(B-1)z-Bz^2}$$

at the point $z = -r$ for $R_1 \le R_2'$ and at the point $z = re^{i\theta}$ for $R_2' \le R_1$, where θ is determined from the equation

$$\mathrm{Re}\{\frac{1+b(A-1)re^{i\theta} - Ar^2e^{2i\theta}}{1+b(B-1)re^{i\theta} - Br^2e^{2i\theta}}\} = R_1 .$$

3.3 Two subclasses of univalent functions with fixed second coefficient

We first establish certain distortion properties for the class $S_b^*(A, B)$. These refine several results obtained previously by Janowski [33] on the class $S^*(A, B)$. We shall also give some simple applications to illustrate, in a sense, the significance of functions with pre-assigned second coefficient.

3.3.1 Theorem. *Let* $f(z) \in S_b^*(A, B)$; *then on* $|z| = r < 1$,

$$rG(r) \le |f(z)| \le rH(r)$$

$$\frac{1+b(1-A)r-Ar^2}{1+b(1-B)r-Br^2} . G(r) \le |f'(z)| \le \frac{1+b(1+A)r+Ar^2}{1+b(1+B)r+Br^2} . H(r)$$

where

$$H(r) = \begin{cases} \exp\{H_1(r;\ A,\ B)\}\ , & \textit{for } B < 0 \textit{ or } \{B > 0 \textit{ and } b^2 \geq 4B/(1+B)^2\}\ , \\ \exp\{H_2(r;\ A,\ B)\}\ , & \textit{for } B > 0 \textit{ and } b^2 \leq 4B/(1+B)^2\ , \\ \exp\{A[\frac{r}{b} + (1-\frac{1}{b^2})\log(1+br)]\}\ , & \textit{for } B = 0 \textit{ and } b \neq 0\ , \\ \exp\{\frac{1}{2}Ar^2\}, & \textit{for } B = 0 \textit{ and } b = 0\ ; \end{cases}$$

$$G(r) = \begin{cases} \exp\{H_1(r;\ -A,\ -B)\}, & \textit{for } B > 0 \textit{ or } \{B < 0 \textit{ and } b^2 \geq -4B/(1-B)^2\}\ , \\ \exp\{H_2(r;\ -A,\ -B)\}, & \textit{for } B < 0 \textit{ and } b^2 \leq -4B/(1-B)^2\ , \\ \exp\{-A[\frac{r}{b} + (1-\frac{1}{b^2})\log(1+br)]\}\ , & \textit{for } B = 0 \textit{ and } b \neq 0\ , \\ \exp\{-\frac{1}{2}Ar^2\}\ , & \textit{for } B = 0 \textit{ and } b = 0\ ; \end{cases}$$

$$H_1(r;\ A,\ B) = \frac{A-B}{2B}\log(1+b(1+B)r+Br^2)$$
$$+\frac{(A-B)(1-B)b}{4B^2r\sqrt{-c_1}}\log\left|\frac{b(1+B)+2Br(1+\sqrt{-c_1})}{b(1+B)+2Br(1-\sqrt{-c_1})}\cdot\frac{b(1+B)-2Br\sqrt{-c_1}}{b(1+B)+2Br\sqrt{-c_1}}\right|\ ,$$

$$H_2(r;\ A,\ B) = \frac{A-B}{2B}\log(1+b(1+B)r+Br^2)$$
$$-\frac{(A-B)(1-B)b}{2B^2r\sqrt{c_1}}\left[\tan^{-1}\left(\frac{2Br+b(1+B)}{2Br\sqrt{c_1}}\right) - \tan^{-1}\left(\frac{b(1+B)}{2Br\sqrt{c_1}}\right)\right]\ ,$$

$$c_1 = \frac{1}{Br^2} - \left[\frac{b(1+B)}{2Br}\right]^2\ .$$

Proof. The structural formula for the class $S_b^*(A,\ B)$ is (see (0.2.10))

$$f(z) = z\exp\int_0^z \frac{p(\xi)-1}{\xi}\,d\xi\ ,\quad p(z) \in P_b(A,\ B)\ .$$

Hence

$$\left|\frac{f(z)}{z}\right| = \exp \operatorname{Re}\left\{ \int_0^z \frac{p(\xi)-1}{\xi}\, d\xi \right\} .$$

Substituting ξ by zt in the integral we get

$$(3.3.1) \qquad \left|\frac{f(z)}{z}\right| = \exp \int_0^1 \operatorname{Re}\left\{\frac{p(zt)-1}{t}\right\} dt .$$

An application of (3.2.6) yields, on $|zt| = rt$,

$$\operatorname{Re}\left\{\frac{p(zt)-1}{t}\right\} \geq -(A-B) \frac{br + r^2 t}{1+b(1-B)rt-Br^2t^2} .$$

Replacing this bound into (3.3.1) and carrying out the integration will give the lower bound for $|f(z)|$. The upper bound may be obtained similarly. From the definition of $S_b^*(A, B)$ we have

$$(3.3.2) \qquad |f'(z)| = \left|\frac{f(z)}{z}\right| |p(z)| , \; p(z) \in P_b(A, B) , \; z \in \Delta .$$

Hence making use of the bounds derived above for $|f(z)|$ together with inequalities (3.2.6), we obtain the corresponding bounds for $|f'(z)|$.

The lower bounds for $|f(z)|$ and $|f'(z)|$ are sharp for the function

$$f(z) = z \exp \int_0^z \frac{(A-B)(b-\xi)}{1+b(B-1)\xi-B\xi^2}\, d\xi ,$$

while their upper bounds are attained for the function

$$f(z) = z \exp \int_0^z \frac{(A-B)(b+\xi)}{1+b(1+B)\xi+B\xi^2}\, d\xi .$$

3.3.2 Remark. For an application of the above theorem, let us consider the function $g(z) = 1/z + b_1 z + b_2 z^2 + \ldots$ which maps the unit disc onto a domain whose complement is starlike with respect to the origin. Then the function $f(z)$ defined by $f(z) = 1/g(z)$, $z \in \Delta$, is starlike in Δ and has the series expansion

$$f(z) = z + a_3 z^3 + a_4 z^4 + \ldots$$

Hence Theorem 3.3.1 with $A = 1$, $B = -1$, $b = 0$ gives

$$\frac{1}{r} - r \le |g(z)| = \frac{1}{|f(z)|} \le \frac{1}{r} + r \ , \ |z| = r \ .$$

Equalities occur for the function $g(z) = 1/z + \epsilon z$, $|\epsilon| = 1$.

As another application of Theorem 3.3.1, we consider the odd starlike functions $f(z)$ of order α, $0 \le \alpha < 1$. Concerning these functions, Robertson [73] proved

$$\frac{r}{(1+r^2)^{1-\alpha}} \le |f(z)| \le \frac{r}{(1-r^2)^{1-\alpha}} \ , \quad |z| = r \ .$$

Putting $A = 1-2\alpha$, $B = -1$, $b = 0$ in Theorem 3.3.1, these results are recovered.

We now obtain the radius of convexity for the class $S_b^*(A, B)$.

3.3.3 Theorem. *The radius of convexity of* $S_b^*(A, B)$ *is given by the smallest root in* $(0, 1]$ *of*

(i) $A^2r^4+b(2A^2-3A+B)r^3+[b^2(1-A)^2-4A+2B]r^2+b(2+B-3A)r+1 = 0$, *for* $R_1 \le R_2'$,

(ii) $(4A^2-5A+B)r^4-2(2A^2-3A+2-B)r^2+4-5A+B = 0$, *for* $R_2' \le R_1$, *where* R_1, R_2' *are as given in Theorem* 3.2.1 *with* $\alpha = \beta = 1$.

Proof. For $f(z) \in S_b^*(A, B)$, we may write

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)} \quad ,$$

for some $p(z) \in P_b(A, B)$. Thus an application of Theorem 3.2.1 with $\alpha = \beta = 1$ yields immediately the equations giving the radius of convexity of $S_b^*(A, B)$. The result is sharp for the function $f_0(z)$ determined from $zf_0'(z)/f_0(z) = p(z)$, where $p(z)$ is extremal for Theorem 3.2.1.

Theorem 3 of McCarty [51] corresponds to the case $A = 1-2\alpha$, $B = -1$. We note that the two bounds in Theorem 3.2.1 are attained by the same function at two different points. Thus the function $f_0(z)$ defined above serves as an extremal function for both cases of Theorem 3.3.3. The second extremal function given by McCarty [51, Theorem 3], in fact, does not belong to the class.

We next consider the class $P_b'(A, B)$. The results on functions whose derivative has a positive real part in the unit disc can be traced back to as early as 1915 with the work of Alexander [3]. The most significant fact about these functions is that they are all univalent in Δ. This follows from Wolff-Noshiro-Warschawski's theorem that if $f(z)$ is regular and satisfies $\mathrm{Re}\{f'(z)\} > 0$ in a convex domain, then it is univalent there.

We denote by P' the class of functions $f(z) \in N$ for which $\mathrm{Re}\{f'(z)\} > 0$ in Δ. Standard properties of P' such as distortion, covering, convexity and coefficient bounds were thoroughly investigated by MacGregor [43]. Recently, McCarty [50] extended Mac Gregor's results to the class $P_b'(1-2\alpha, -1)$ of functions $f(z)$ with fixed second coefficient $2b(1-\alpha)$ and satisfying $\mathrm{Re}\{f'(z)\} > \alpha$, $0 \le \alpha < 1$, $z \in \Delta$.

In the following, we generalise the results by McCarty concerning distortion bounds and the radius of convexity of $P_b'(1-2\alpha, -1)$ to the class $P_b'(A, B)$.

3.3.4 <u>Theorem</u>. *Let* $f(z) \in P_b'(A, B)$; *then on* $|z| = r < 1$,

$$\frac{1+b(1-A)r-Ar^2}{1+b(1-B)r-Br^2} \le \mathrm{Re}\{f'(z)\} \le |f'(z)| \le \frac{1+b(1+A)r+Ar^2}{1+b(1+B)r+Br^2} ;$$

$$|f(z)| \le \begin{cases} G_1(r;\, A,\, B), \ \textit{for } B < 0 \textit{ or } \{B > 0 \textit{ and } b^2 \ge 4B/(1+B)^2\} , \\ G_2(r;\, A,\, B), \ \textit{for } B > 0 \textit{ and } b^2 \le 4B/(1+B)^2 , \\ \dfrac{Ar^2}{2b} + (1+A - \dfrac{A}{b^2})r + \dfrac{A(1-b^2)}{b^3}\log(1+br) , \ \textit{for } B = 0,\ b \ne 0 , \\ r + Ar^3/3 , \ \textit{for } B = 0 ,\ b = 0 ; \end{cases}$$

$$|f(z)| \ge \begin{cases} G_1(r;\, -A,\, -B) , \ \textit{for } B > 0 \textit{ or } \{B < 0 \textit{ and } b^2 \ge -4B/(1-B)^2\} , \\ G_2(r;\, -A,\, -B) , \ \textit{for } B < 0 \textit{ and } b^2 \le -4B/(1-B)^2 , \\ -\dfrac{Ar^2}{2b} + (1-A + \dfrac{A}{b^2})r - \dfrac{A(1-b^2)}{b^3}\log(1+br), \ \textit{for } B = 0,\ b \ne 0 , \\ r - Ar^3/3 , \ \textit{for } B = 0 ,\ b = 0 ; \end{cases}$$

where

$$G_1(r;\ A,\ B)=\frac{Ar}{B} - \frac{b(A-B)}{2B^2}\log(1+b(1+B)r+Br^2)$$

$$+ \frac{A-B}{2B^2}[1- \frac{b^2(1+B)}{2B}]\frac{1}{\sqrt{-c_2}}\ \log\left|\frac{2Br+b(1+B)+2B\sqrt{-c_2}}{2Br+b(1+B)-2B\sqrt{-c_2}}.\ \frac{b(1+B)-2B\sqrt{-c_2}}{b(1+B)+2B\sqrt{-c_2}}\right|\ ,$$

$$G_2(r;\ A,\ B)=\frac{Ar}{B} - \frac{b(A-B)}{2B^2}\log(1+b(1+B)r+Br^2)$$

$$- \frac{A-B}{B^2}[1- \frac{b^2(1+B)}{2B}]\frac{1}{\sqrt{c_2}}[\tan^{-1}(\frac{2Br+b(1+B)}{2B\sqrt{c_2}})- \tan^{-1}\frac{b(1+B)}{2B\sqrt{c_2}}\ ]\ ,$$

$$c_2 = \frac{1}{B} - [\frac{b(1+B)}{2B}]^2\ .$$

<u>Proof</u>. Since $f'(z) \in P_b(A, B)$, the bounds for $\mathrm{Re}\{f'(z)\}$ and $|f'(z)|$ follow immediately from (3.2.6). The bounds for $|f(z)|$ are derived from the fact that

$$f(z) = \int_0^z f'(\xi)\ d\xi = \int_0^{|z|} f'(te^{i\theta})\ e^{i\theta}dt\ .$$

Thus, on $|z| = r$,

$$|f(z)| \le \int_0^r |f'(te^{i\theta})|dt \le \int_0^r \frac{1+b(1+A)t+At^2}{1+b(1+B)t+Bt^2}\ dt,$$

$$|f(z)| \ge \int_0^r \mathrm{Re}\{f'(te^{i\theta})\}dt \ge \int_0^r \frac{1+b(1-A)t-At^2}{1+b(1-B)t-Bt^2}\ dt\ .$$

Carrying out the integration we get the bounds for $|f(z)|$.

The upper bounds for $|f'(z)|$ and $|f(z)|$ are attained for the function

$$f(z) = \int_0^z \frac{1+b(1+A)\xi+A\xi^2}{1+b(1+B)\xi+B\xi^2}\ d\xi \qquad \text{at } z = r\ ,$$

while the lower bounds for $\operatorname{Re}\{f'(z)\}$ and $|f(z)|$ are attained for the function

$$f(z) = \int_0^z \frac{1+b(A-1)\xi-A\xi^2}{1+b(B-1)\xi-B\xi^2}\,d\xi \quad \text{at } z = -r .$$

For $f(z) \in P_b'(A, B)$, we have

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zp'(z)}{p(z)} , \quad z \in \Delta$$

for some $p(z) \in P_b(A, B)$. Thus an application of Theorem 3.2.1 with $\alpha = 0$, $\beta = 1$ gives

3.3.5 __Theorem__. *The radius of convexity of* $P_b'(A, B)$ *is given by the smallest root in* $(0, 1]$ *of*

(i) $ABr^4-2bA(1-B)r^3+[b^2(1-A)(1-B)+B-3A]r^2+2b(1-A)r+1 = 0$, *for* $R_1 \le R_2'$,

(ii) $A(1-B)r^4+(1-A)(1-B)r^2 - (1-A) = 0$, *for* $R_2' \le R_1$,

where R_1, R_2' *are as given in Theorem* 3.2.1 *with* $\alpha = 0$, $\beta = 1$.

The result is sharp for the function $f_1(z) = \int_0^z p(\xi)d\xi$, where $p(z)$ is extremal for Theorem 3.2.1.

Putting $A = 1-2\alpha$, $B = -1$, we obtain Theorem 2 of McCarty [51]. Again here, we remark that the function $f_1(z)$ defined above is extremal for both cases of Theorem 3.3.5. The second extremal function given by McCarty [51, Theorem 2], in fact, does not belong to the class.

3.4 The radius of convexity of a subclass of close-to-starlike functions with fixed second coefficient

We denote by $\mathcal{R}$ the class of functions $f(z) \in \mathcal{N}$ which satisfy $\mathrm{Re}\{f(z)/z\} > 0$ in Δ. Mac Gregor [45] proved that the radius of starlikeness of this class is $\sqrt{2} - 1$. Furthermore, Reade, Ogawa and Sakaguchi [72] showed that the radius of convexity of $\mathcal{R}$ is given by the smallest positive root of the equation

$$1 - 5r - 3r^2 - r^3 = 0 .$$

Chichra [15] obtained the corresponding result for k-fold symmetric functions in $\mathcal{R}$.

In this section, we determine the radius of convexity for functions in $\mathcal{R}$ with a fixed negative second coefficient, that is, for the class

$$\mathcal{R}_b = \{f(z) = z - 2bz^2 + \ldots;\ 0 \le b \le 1,\ \mathrm{Re}\{f(z)/z\} > 0\ ,\ z \in \Delta\} .$$

In fact, we shall establish the radius of convexity for the family of functions $f(z) = z - 2bz^2 + \ldots$ which satisfy

$$(3.4.2) \qquad \left|\frac{f(z)}{z} - \alpha\right| < \alpha\ \ ,\ \ \alpha \ge 1\ ,\ z \in \Delta$$

and $0 \le b \le 1 - 1/2\alpha$. Letting $\alpha \to \infty$ in this result we obtain the radius of convexity of $\mathcal{R}_b$.

We remark that there is no essential restriction in assuming the second coefficient of the functions $f(z)$ be real and negative, for, if this is not the case, we may consider the functions $e^{i\theta}f(e^{-i\theta}z) = z - |a_2|z^2 + \ldots$, where $\theta = \arg a_2 + \pi$. It is clear from equation (3.4.5) below that this assumption is convenient for our purpose. By this equation, every such function $f(z)$ can be represented in terms of functions of positive real part with a fixed positive first coefficient and thus, we can make use of properties of these functions, which have already been established in Section 3.2 .

Let $f(z) = z - 2bz^2 + \ldots$ be characterised by (3.4.2). Then putting

$$\psi(z) = 1 - \frac{1}{\alpha}\frac{f(z)}{z} \ , \tag{3.4.3}$$

we have $|\psi(z)| < 1$ and $\psi(0) = 1 - 1/\alpha$. Thus the function $w(z)$ defined by

$$w(z) = \frac{\psi(z) - \psi(0)}{1 - \psi(0)\psi(z)} \ , \quad z \in \Delta \ , \tag{3.4.4}$$

is in $\mathcal{B}$. From (3.4.3) and (3.4.4) we deduce

$$f(z) = z \cdot \frac{1 - w(z)}{1+(1-1/\alpha)w(z)} \ , \quad z \in \Delta \ .$$

Since $w(z) = [p(z)-1]/[p(z)+1]$ for some $p(z) \in \mathcal{P}$, the function $f(z)$ can be rewritten as

$$f(z) = \frac{2\alpha z}{1+(2\alpha-1)p(z)} \ , \quad z \in \Delta \ . \tag{3.4.5}$$

From the power series expansion of $f(z)$ and (3.4.5) we have that $p(z) = 1 + 2tz + \ldots$, where $t = 2\alpha b/(2\alpha-1)$. Also, $0 \le t \le 1$ as $0 \le b \le 1-1/2\alpha$. Thus $p(z)$ belongs to $\mathcal{P}_t(1, -1) \equiv \mathcal{P}_t$, the class of functions of positive real part with first coefficient equal to $2t$.

The representation (3.4.5) yields

$$(3.4.6)\qquad 1+\frac{zf''(z)}{f'(z)} = 1 - \frac{2zp'(z)}{1/(2\alpha-1)+p(z)} - \frac{z^2p''(z)}{1/(2\alpha-1)+p(z)-zp'(z)} .$$

Consequently, in order to obtain the radius of convexity of f(z), we require the upper bounds for $|zp'(z)/(\mu+p(z)|$ and $|z^2p''(z)/(\mu+p(z) - zp'(z))|$ on $|z| = r < 1$, where $\mu = 1/(2\alpha-1)$ and p(z) varies in P_t. These are established in the following lemmas.

3.4.1 Lemma. *If* $p(z) \in P_t$, $\mu > 0$, *then on* $|z| = r < 1$,

$$(3.4.7)\qquad \left|\frac{p(z)}{p(z)+\mu}\right| \le \frac{1+2tr+r^2}{1+\mu+2tr+(1-\mu)r^2} .$$

Proof. Since $p(z) = [1+w(z)]/[1-w(z)]$ for $w(z) = tz + \ldots \in B$, we have

$$\frac{p(z)}{p(z)+\mu} = \frac{1}{1+\mu}\cdot\frac{1+w(z)}{1+[(1-\mu)/(1+\mu)]w(z)} .$$

Also, $-1 < (1-\mu)/(1+\mu) < 1$ for $\mu > 0$. Hence the assertion follows from (3.2.6) with $A = 1$, $B = (1-\mu)/(1+\mu)$ and b replaced by t.

Equality in (3.4.7) occurs for the function $p(z) = (1+2tz+z^2)/(1-z^2)$ at $z = r$.

3.4.2 Lemma. *If* $p(z) \in P_t$, *then for* $|z| < 1$,

$$(3.4.8)\qquad |p'(z)| \le 2\frac{\mathrm{Re}\{p(z)\}}{1-|z|^2}\cdot\frac{t+2|z|+t|z|^2}{1+2t|z|+|z|^2} .$$

Proof. Write $p(z) = [1+z\psi(z)]/[1-z\psi(z)]$, where $\psi(z) = t + \ldots$ is such that $|\psi(z)| \le 1$ in Δ . Then, from (3.2.2) ,

$$(3.4.9) \qquad |\psi(z)| \le \frac{|z| + t}{1 + t|z|} \quad .$$

Also,

$$|p'(z)| = \frac{2[z\psi'(z)+\psi(z)]}{[1-z\psi(z)]^2} \quad , \quad z \in \Delta \quad .$$

Thus,

$$|p'(z)| = \frac{2|z\psi'(z)+\psi(z)|}{1-|z\psi(z)|^2} \cdot \frac{1-|z\psi(z)|^2}{|1-z\psi(z)|^2}$$

$$= \frac{2|z\psi'(z)+\psi(z)|}{1-|z\psi(z)|^2} \cdot \mathrm{Re}\{p(z)\}$$

$$\le 2\,\mathrm{Re}\{p(z)\} \cdot \frac{|z\psi'(z)|+|\psi(z)|}{1-|z\psi(z)|^2}$$

$$\le 2\,\mathrm{Re}\{p(z)\} \cdot \frac{|z|(1-|\psi(z)|^2)/(1-|z|^2)+|\psi(z)|}{1-|z\psi(z)|^2} \quad , \text{ from (1.2.5)}$$

$$(3.4.10) \qquad = \frac{2\,\mathrm{Re}\{p(z)\}}{1-|z|^2} \cdot \frac{|\psi(z)|+|z|}{1+|z|\,|\psi(z)|} \quad .$$

The function $(|\psi(z)| + |z|)/(1+|z|\,|\psi(z)|)$ is monotonically increasing with respect to $|\psi(z)|$; hence from (3.4.9) and (3.4.10) the result follows.

3.4.3 Lemma. *If* $p(z) \in P_t$, $\mu > 0$, *then on* $|z| = r < 1$,

$$(3.4.11) \qquad \left|\frac{zp'(z)}{p(z)+\mu}\right| \le \frac{2r}{1-r^2} \cdot \frac{t+2r+tr^2}{1+\mu+2tr+(1-\mu)r^2} \quad .$$

Proof. For $\mu > 0$, we have

$$\left|\frac{zp'(z)}{p(z)+\mu}\right| \le \frac{|zp'(z)|}{\mathrm{Re}\{p(z)\}+\mu} = \frac{|zp'(z)|}{\mathrm{Re}\{p(z)\}} \cdot \frac{1}{1+\mu/\mathrm{Re}\{p(z)\}}$$

$$\le \frac{2r}{1-r^2} \cdot \frac{t+2r+tr^2}{1+2tr+r^2} \cdot \left(1+\frac{\mu(1-r^2)}{1+2tr+r^2}\right)^{-1} \text{, from (3.4.8) and (3.2.6)}$$

$$= \frac{2r}{1-r^2} \cdot \frac{t+2r+tr^2}{1+\mu+2tr+(1-\mu)r^2} .$$

Equality in (3.4.11) is attained for the function $p(z) = (1+2tz+z^2)/(1-z^2)$ at $z = r$.

The next lemma establishes an inequality which involves the second derivative of $p(z)$. This is based on the well-known result that if $p(z) = 1 + p_1 z + p_2 z^2 + \ldots \in P$, then $|p_2| \le 2$. The bound also holds true for functions in P with a fixed first coefficient. Indeed, let $p(z) = 1 + 2tz + p_2 z^2 + \ldots \in P_t$, $0 \le t \le 1$, then from the representation $p(z) = [1+z\psi(z)]/[1-z\psi(z)]$, where $\psi(z) = t + \ldots$ and satisfies $|\psi(z)| \le 1$ in Δ , we get after equating the coefficients of same powers of z,

(3.4.12) $$2b_1 = p_2 - 2t^2 .$$

It follows from Carathéodory's inequality (1.2.5) that

$$b_1 \le 1 - |t|^2 .$$

Thus, in view of (3.4.12) , we have

$$|p_2 - 2t^2| \le 2 - 2t^2 ,$$

that is, $|p_2| \le 2$, which is sharp for the function

$$p(z) = \frac{1+2tz+z^2}{1 - z^2} = 1 + 2tz + 2z^2 + \ldots$$

Now, let ξ be a complex number such that $0 < |\xi| < 1$ and $p(z) \in P_t$. Then the function $q(z)$ defined by

$$q(z) = p(\frac{z+\xi}{1+\bar{\xi}z}) = p(\xi)+(1-|\xi|^2)p'(\xi)z+\tfrac{1}{2}(1-|\xi|^2)[(1-|\xi|^2)p''(\xi)-2\bar{\xi}p'(\xi)]z^2+\ldots$$

is regular and satisfies $\operatorname{Re}\{q(z)\} > 0$ in Δ . Hence from the above remark, the following lemma follows.

3.4.4 <u>Lemma</u>. *If* $p(z) \in P_t$, *then for* $|z| < 1$,

$$(3.4.13) \qquad |zp''(z) - \frac{2|z|^2}{1-|z|^2} p'(z)| \le \frac{4|z|}{(1-|z|^2)^2} \quad |p(z)| .$$

In view of inequality (3.4.13) we get for $|z| < 1$

$$\left|\frac{z^2p''(z)}{p(z)+\mu}\right| \le \frac{2|z|^2}{1-|z|^2} \left|\frac{zp'(z)}{p(z)+\mu}\right| + \frac{4|z|^2}{(1-|z|^2)^2} \left|\frac{p(z)}{p(z)+\mu}\right| .$$

Thus an application of (3.4.11) and (3.4.7) to the right-hand side yields

3.4.5 <u>Lemma</u>. *If* $p(z) \in P_t$, $\mu > 0$, *then on* $|z| = r < 1$,

$$(3.4.14) \qquad \left|\frac{z^2p''(z)}{p(z)+\mu}\right| \le \frac{4r^2(1+3tr+3r^2+tr^3)}{(1-r^2)^2[1+\mu+2tr+(1-\mu)r^2]} .$$

Equality occurs for the function $p(z) = (1+2tz+z^2)/(1-z^2)$ at $z = r$.

We are now in a position to prove the main result of this section.

3.4.6 Theorem. *Let* $f(z) = z - 2bz^2 + \ldots$ *be regular in* Δ *and satisfy*

$$\left|\frac{f(z)}{z} - \alpha\right| < \alpha, \quad \alpha \ge 1 \ , \ z \in \Delta$$

and $0 \le b \le 1-1/2\alpha$. *Put* $t = 2\alpha b/(2\alpha-1)$ *and* $\mu = 1/(2\alpha-1)$. *Then the radius of convexity of* $f(z)$ *is given by the smallest root in* $(0,1]$ *of the equation*

$$(1+\mu)^2 - 2t(1+\mu)r - 3(1+\mu)(5+\mu)r^2 - 4t(6+5\mu)r^3 - (1+2\mu-3\mu^2+16t^2)r^4$$

$$-6t(1-\mu)r^5 - (1-\mu)^2r^6 = 0 \ .$$

Proof. As derived earlier in (3.4.6)

$$1 + \frac{zf''(z)}{f'(z)} = 1 - \frac{2zp'(z)}{p(z)+\mu} - \frac{z^2p''(z)}{p(z)+\mu-zp'(z)} \quad ,$$

where $p(z) \in P_t$. Hence

$$(3.4.15) \qquad \mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\} \ge 1 - \left|\frac{2zp'(z)}{p(z)+\mu}\right| - \left|\frac{z^2p''(z)}{p(z)+\mu}\right|\left|1 - \frac{zp'(z)}{p(z)+\mu}\right|^{-1} .$$

Now, from (3.4.11), we have

$$\left|1 - \frac{zp'(z)}{p(z)+\mu}\right| \ge 1 - \left|\frac{zp'(z)}{p(z)+\mu}\right|$$

$$\ge 1 - \frac{2r}{1-r^2} \cdot \frac{t+2r+tr^2}{1+\mu+2tr+(1-\mu)r^2}$$

$$(3.4.16) \qquad = \frac{1+\mu-2(2+\mu)r^2-4tr^3-(1-\mu)r^4}{(1-r^2)[1+\mu+2tr+(1-\mu)r^2]} \quad .$$

It is easy to check that the numerator has a root in (0, 1). Let σ be its smallest root in (0, 1); then for $|z| < \sigma$, we obtain

$$(3.4.17) \quad \left|1 - \frac{zp'(z)}{p(z)+\mu}\right|^{-1} \le \frac{(1-r^2)[1+\mu+2tr+(1-\mu)r^2]}{1+\mu-2(2+\mu)r^2-4tr^3-(1-\mu)r^4} \quad .$$

Applying the bounds given by (3.4.11), (3.4.14) and (3.4.17) to (3.4.15) we get

$$\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \ge \frac{F(r)}{[1+\mu+2tr+(1-\mu)r^2][1+\mu-2(2+\mu)r^2-4tr^3-(1-\mu)r^4]} \quad ,$$

where

$$F(r) = (1+\mu)^2 - 2t(1+\mu)r - 3(1+\mu)(5+\mu)r^2 - 4t(6+5\mu)r^3$$

$$- (1+2\mu-3\mu^2+16t^2)r^4 - 6t(1-\mu)r^5 - (1-\mu)^2r^6 \quad .$$

Since $F(0) = (1+\mu)^2 > 0$, $F(1) = -16-16\mu-16t\mu-32t-16t^2 < 0$, $F(r)$ has a root in (0, 1). Denote its smallest root in (0, 1) by ρ ; then the condition $\mathrm{Re}\{1+zf''(z)/f'(z)\} > 0$ is satisfied in $|z| < \min(\rho, \sigma)$. We further note that, for $f(z)$ as defined,

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} = \mathrm{Re}\{1 - \frac{zp'(z)}{p(z)+\mu}\} \ge 1 - \left|\frac{zp'(z)}{p(z)+\mu}\right| \quad .$$

Thus, in view of (3.4.16), we have $\mathrm{Re}\{zf'(z)/f(z)\} > 0$ in $|z| < \sigma$. In other words, $f(z)$ is starlike in $|z| < \sigma$. Since the radius of starlikeness of $f(z)$ is greater than or equal to its radius of convexity, we get $\rho \le \sigma$ and the assertion follows.

To see that the result is sharp, we consider the function

$$f(z) = \frac{\alpha z(1 - z^2)}{\alpha+(2\alpha-1)tz+(\alpha-1)z^2} \quad .$$

The case $b = 1$, $\alpha \to \infty$ corresponds to the theorem of Reade, Ogawa and Sakaguchi [72] .

CHAPTER 4

REGULAR FUNCTIONS WITH A CONSTRAINT

4.1 Introduction

In this chapter we study the following subclass of the class of functions of positive real part in the unit disc:

$$\mathcal{P}[a, b] = \{p(z) \in \mathcal{P} \; ; \; p(a) = b, \; 0 < a < 1 \; , \; b > 0\} \; .$$

Since $\mathcal{P}$ is invariant with respect to rotations of Δ, there is no loss of generality when we assume $0 < a < 1$.

Interest in the class $\mathcal{P}[a, b]$ arises from the recent investigations of the family $\Sigma^c_{(s)}$ of meromorphic convex functions with a simple pole at $z = s$, $0 < s < 1$, by Royster [79] and Pfaltzgraff and Pinchuk [64]. In fact, the latter authors gave an analytic characterisation for the class $\Sigma^c_{(s)}$ in terms of functions of $\mathcal{P}[a, b]$ as follows.

Let $a = s$, $b = (1+s^2)/(1-s^2)$; then a necessary and sufficient condition for a univalent function $f(z)$ with a pole at $z = s$, $0 < s < 1$, and normalised by $f(0) = 0$, $f'(0) = 1$ to map the unit disc onto the exterior of a convex domain is

$$(4.1.1) \qquad 1 + \frac{zf''(z)}{f'(z)} + \frac{z+s}{z-s} - \frac{1+sz}{1-sz} = - p(z) \quad , \; z \in \Delta \quad ,$$

for some $p(z) \in \mathcal{P}[a, b]$.

In this chapter we are interested in functionals of the form

$$\mathrm{Re}\{F(p(z), zp'(z))\} \quad , \quad p(z) \in \mathcal{P}[a, b] \; ,$$

where $F(u, v)$ is regular in the v-plane and in the half-plane $\text{Re } u > 0$. By deriving a representation formula for $P[a, b]$ in terms of functions in P and using the Golusin's variational method (see (0.1.17) and (0.1.18)), Pfaltzgraff and Pinchuk [64] showed that the value of

$$(4.1.2) \qquad \min_{p(z) \in P[a,\, b]} \; \min_{|z|=r<1} \text{Re}\{F(p(z), zp'(z))\}$$

occurs only for functions of the form

$$p(z) = \sum_{k=1}^{n+1} \beta_k (e^{it_k} + z)/(e^{it_k} - z) \ ,$$

where $n \le 2$, $\beta_k \ge 0$, $\sum_{k=1}^{n+1} \beta_k = 1$ and $\sum_{k=1}^{n+1} \beta_k (e^{it_k} + a)/(e^{it_k} - a) = b$.
However, using the above result, the problem (4.1.2) which involves the determination of the parameters ψ, β_k, t_k, where $z = re^{i\psi}$, $k \le 3$, can only be solved when $F(u, v)$ is of a relatively simple form.

Pfaltzgraff and Pinchuk [64] obtained the lower bound on $|z| = r$ for $\text{Re}\{p(z)\}$ over $P[a, b]$ in the case $a = s$, $b = (1+s^2)/(1-s^2)$. As far as we are aware of, no other result on $P[a, b]$ is known.

In this chapter we establish best possible bounds for $|p'(z)|$, $|zp'(z)/p(z)|$, $\text{Re}\{zp'(z)/p(z)\}$ and $\text{Re}\{p(z)+zp'(z)\}$ over $P[a, b]$ when $b > 1$. We note that the restriction $b > 1$ is sufficient for the applications to the class $\sum^{c}_{(s)}$ and related subclasses, for then we have $b>1$
$\Leftrightarrow 0<s^2<1$
$\cancel{b = (1+s^2)/(1-s^2) > 1}$. To solve these problems, we represent the functions $p(z) \in P[a, b]$ in terms of functions of B and make use of Dieudonné's lemma.

The results on $P[a, b]$ will then be used to derive certain distortion properties for two subclasses of regular functions associated with $P[a, b]$, namely,

$$R[a, b] = \{f(z) \in N \; ; \; f(z)/z \in P[a, b] \; , \; z \in \Delta\} \; ,$$

$$S^{*}[a, b] = \{f(z) \in N \; ; \; zf'(z)/f(z) \in P[a, b] \; , \; z \in \Delta\} \; .$$

These refine the corresponding results for the two well-known subclasses R and S^{*}. The distortion bounds for $S^{*}[a, b]$ are further employed to establish estimates on $|f'(z)|$, where $f(z)$ varies in $\sum_{(S)}^{c}$.

4.2 Representation formulae for $P[a, b]$

Let $p(z) \in P[a, b]$; then since $p(z)$ is also a function in P , we have that

$$\frac{1 - a}{1 + a} \leq |p(z)| \leq \frac{1 + a}{1 - a}$$

for $|z| \leq a < 1$. Consequently, from the condition $p(a) = b$, b satisfies the inequalities

$$\frac{1 - a}{1 + a} \leq b \leq \frac{1 + a}{1 - a} \quad . \tag{4.2.1}$$

Also, the normalisation of P shows that $b = 1$ only when $a = 0$; hence we shall not consider the case $b = 1$ in the following.

We first look at the case $b < 1$. Denote by E the set of functions $\psi(z)$ regular in Δ and such that $|\psi(z)| \leq 1$ there. Define

$$T(z) = \frac{a - z}{1 - az} \quad , \quad z \in \Delta$$

and put $A = (1-b)/a(1+b)$. Then it follows from (4.2.1) that $0 < A \le 1$. For $p(z) \in \mathcal{P}[a, b]$, the function

$$\psi_1(z) = \frac{1}{z} \cdot \frac{1 - p(z)}{1 + p(z)} \tag{4.2.2}$$

is in $\mathcal{E}$ and $\psi_1(a) = A$. Hence the function

$$\psi_2(z) = \frac{\psi_1(z) - A}{1 - A\psi_1(z)} \tag{4.2.3}$$

is in $\mathcal{E}$ and $\psi_2(a) = 0$, from which it follows that the function

$$\chi(z) = \frac{\psi_2(z)}{T(z)} \tag{4.2.4}$$

belongs to $\mathcal{E}$. From (4.2.2), (4.2.3) and (4.2.4) we deduce a representation formula for $\mathcal{P}[a, b]$ in terms of functions of $\mathcal{B}$ in the case $b < 1$ as

$$p(z) = \frac{1 - w_1(z)}{1 + w_1(z)} \quad , \quad z \in \Delta , \tag{4.2.5}$$

where

$$w_1(z) = z\,\frac{A + T(z)\chi(z)}{1 + AT(z)\chi(z)} \in \mathcal{B} .$$

We next consider the case $b > 1$. Putting $B = (b-1)/a(b+1)$, then it is clear from (4.2.1) that $0 < B \le 1$. For $p(z) \in \mathcal{P}[a, b]$, we define

$$\psi_3(z) = \frac{1}{z}\,\frac{p(z)-1}{p(z)+1} \quad , \tag{4.2.6}$$

then $\psi_3(z) \in \mathcal{E}$ and $\psi_3(a) = B$. The function

(4.2.7) $$\psi_4(z) = \frac{\psi_3(z) - B}{1 - B\psi_3(z)}$$

is therefore in E and $\psi_4(a) = 0$. Hence the function

(4.2.8) $$\psi(z) = \frac{\psi_4(z)}{T(z)}$$

belongs to E . From (4.2.6), (4.2.7) and (4.2.8) , it follows that a function $p(z) \in P[a, b]$, $b > 1$, can be represented in the form

(4.2.9) $$p(z) = \frac{1 + w_2(z)}{1 - w_2(z)} \quad , \quad z \in \Delta \quad ,$$

where

$$w_2(z) = z\,\frac{B + T(z)\psi(z)}{1 + BT(z)\psi(z)} \in B \quad .$$

In the next section, we derive some distortion bounds for functions in $P[a, b]$. Our starting point is the representation formulae (4.2.5) and (4.2.9). However, we have been unable to combine the two cases to obtain sharp results when b ranges over the complete interval $[(1-a)/(1+a)\ ,\ (1+a)/(1-a)]$.

4.3 Some distortion inequalities for $P[a, b]$

4.3.1 Theorem. *If* $p(z) \in P[a, b]$, *then on* $|z| = r < 1$,

(i) *for* $b < 1$,

(4.3.1) $$\operatorname{Re}\{p(z)\} \geq \frac{1-Dr}{1+Dr} ,$$

(4.3.2) $$|p(z)| \leq \frac{1+Dr}{1-Dr} \quad ;$$

(ii) *for* $b > 1$,

(4.3.3) $$\mathrm{Re}\{p(z)\} \geq \frac{1-Er}{1+Er} ,$$

(4.3.4) $$|p(z)| \leq \frac{1+Er}{1-Er} ,$$

where

$$D = \frac{A+C}{1+AC} , \quad E = \frac{B+C}{1+BC} , \; A = \frac{1-b}{a(1+b)} , \; B = \frac{b-1}{a(b+1)} , \; C = \frac{a+r}{1+ar} .$$

The inequalities (4.3.2) *and* (4.3.3) *are sharp.*

Proof. For the case $b < 1$, $p(z)$ is represented by (4.2.5) with $w_1(z) = z(A+T(z)\chi(z))/(1+AT(z)\chi(z))$. Now $|T(z)| \leq C$ on $|z| = r$. Thus $|T(z)\chi(z)| \leq C$ on $|z| = r$, which yields $|w_1(z)| \leq Dr$ on $|z| = r$. The Subordination Principle (see 0.2.5) shows that the transformation (4.2.5) maps $|z| \leq r$ into the disc

$$|p(z) - \frac{1+D^2r^2}{1-D^2r^2}| \leq \frac{2Dr}{1-D^2r^2} ,$$

from which (4.3.1) and (4.3.2) follow. The inequalities (4.3.3) and (4.3.4) may be similarly obtained by showing that the transformation (4.2.9) maps $|z| \leq r$ into the disc

$$|p(z) - \frac{1+E^2r^2}{1-E^2r^2}| \leq \frac{2Er}{1-E^2r^2} .$$

Inequality (4.3.2) is sharp for the function

$$p_1(z) = \frac{1+AT(z)-z(A+T(z))}{1+AT(z)+z(A+T(z))}$$

at $z = -r$, while inequality (4.3.3) is sharp for the function

$$p_2(z) = \frac{1+BT(z)+z(B+T(z))}{1+BT(z)-z(B+T(z))}$$

at $z = -r$.

With applications and sharpness of results in mind, we shall consider only the case $b > 1$ for the remainder of this chapter.

4.3.2 Theorem. *If* $p(z) \in P[a, b]$, $b>1$ *then on* $|z| = r < 1$,

$$(4.3.5) \qquad |p'(z)| \le \mathrm{Re}\{p(z)\}.\ \frac{2}{1-r^2} \cdot \frac{E+r}{1+Er} ,$$

where E *is as given in Theorem* 4.3.1. *The result is sharp.*

Proof. Write $p(z) = [1+z\Phi(z)]/[1-z\Phi(z)]$, where

$$\Phi(z) = \frac{B+T(z)\psi(z)}{1+BT(z)\psi(z)} ,$$

$T(z)$, $\psi(z)$ being as defined in Section 4.2. Then $|\Phi(z)| \le E$ and

$$p'(z) = \frac{2[z\Phi'(z)+\Phi(z)]}{[1-z\Phi(z)]^2} .$$

Using the same argument as that in the proof of Lemma 3.4.2, we arrive at the inequality

$$|p'(z)| \le \mathrm{Re}\{p(z)\}.\ \frac{2}{1-|z|^2} \cdot \frac{|\Phi(z)| + |z|}{1+|z||\Phi(z)|} .$$

The function $(\Phi(z)+|z|)/(1+|z||\Phi(z)|)$ is monotonically increasing with respect to $|\Phi(z)|$ and $|\Phi(z)| \le E$, hence on $|z| = r$,

$$|p'(z)| \le \text{Re }\{p(z)\} \cdot \frac{2}{1-r^2} \cdot \frac{E+r}{1+Er} \cdot$$

To see that the result is sharp, we consider the function $\Phi_0(z) = [B+T(z)]/[1+BT(z)]$. Then

$$\Phi_0'(z) = \frac{(1-B^2)T'(z)}{[1+BT(z)]^2} \quad , \quad T'(z) = -\frac{1-a^2}{(1-az)^2} \quad .$$

At $z = -r$,

$$|z\Phi_0'(z) + \Phi_0(z)| = \frac{r(1-B^2)(1-a^2)}{(1+BC)^2(1+ar)^2} + E$$

and

$$|z| \frac{1-|\Phi_0(z)|^2}{1-|z|^2} + |\Phi_0(z)| = \frac{r(1-B^2)[1-T^2(-r)]}{(1-r^2)(1+BC)^2} + E$$

$$= \frac{r(1-B^2)(1-a^2)}{(1+BC)^2(1+ar)^2} + E$$

as $1 - T^2(z) = (1-a^2)(1-z^2)/(1-az)^2$. Consequently,

$$|z\Phi_0'(z) + \Phi_0(z)| = |z| \frac{1-|\Phi_0(z)|^2}{1-|z|^2} + |\Phi_0(z)|$$

at $z = -r$. Also, the bound $|\Phi(z)| \le E$ is attained for $\Phi_0(z)$ at $z = -r$. Hence the sharpness of the theorem follows.

As a simple consequence of (4.3.5) we have

4.3.3 <u>Corollary</u>. *If* $p(z) \in P[a, b]$, $b>1$, *then*

(4.3.6) $$|p'(0)| \le 2\,\frac{a+B}{1+aB} \quad ,$$

$$(4.3.7) \qquad \left|\frac{zp'(z)}{p(z)}\right| \le \frac{2r}{1-r^2} \cdot \frac{E+r}{1+Er} \;, \qquad |z| = r < 1 \;,$$

B, E *being as given in Theorem* 4.3.1 .

Both results are sharp for the function

$$p_2(z) = \frac{1+BT(z)+z(B+T(z))}{1+BT(z)-z(B+T(z))} = \frac{(1+aB)(1-z^2)}{1+aB-2(a+B)z+(1+aB)z^2}$$

$$= 1 + 2\,\frac{a+B}{1+aB}\,z + \ldots$$

The lower bound for $\mathrm{Re}\{zp'(z)/p(z)\}$ is less simple to derive. However, making use of Dieudonné's lemma, we can prove

4.3.4 <u>Theorem</u>. *If* $p(z) \in P[a,\, b]$, $b>1$ *then on* $|z| = r < 1$,

$$(4.3.8) \qquad \mathrm{Re}\{\frac{zp'(z)}{p(z)}\} \ge -\frac{2r}{1-r^2} \cdot \frac{E+r}{1+Er} \quad ,$$

E *being as given in Theorem* 4.3.1. *The result is sharp.*

<u>Proof</u>. Write $p(z) = [1+w_2(z)]/[1-w_2(z)]$, $w_2(z)$ being as given by (4.2.9). Then

$$\mathrm{Re}\{\frac{zp'(z)}{p(z)}\} = 2\,\mathrm{Re}\{\frac{zw_2'(z)}{1-w_2(z)^2}\}$$

$$\ge 2[\mathrm{Re}\{\frac{w_2(z)}{1-w_2(z)^2}\} - \frac{r^2-|w_2(z)|^2}{(1-r^2)|1-w_2(z)^2|}] \quad , \text{by } (1.2.5)$$

$$(4.3.9) \qquad = \tfrac{1}{2}\,[\mathrm{Re}\{p(z)-\frac{1}{p(z)}\} - \frac{r^2|p(z)+1|^2-|p(z)-1|^2}{(1-r^2)|p(z)|}] \quad .$$

From the proof of Theorem 4.3.1 we have that the image of $|z| \le r$ under $p(z)$ is contained in the disc $|p(z) - \alpha| \le d$, where

$$\alpha = \frac{1+E^2r^2}{1-E^2r^2} \quad , \qquad d = \frac{2Er}{1-E^2r^2} \quad .$$

Now, put $p(z) = \alpha + u + iv$, $|p(z)| = R$; then (4.3.9) becomes

$$\mathrm{Re}\{\frac{zp'(z)}{p(z)}\} \ge \tfrac{1}{2}\,[\alpha+u-\frac{\alpha+u}{R^2} + \frac{u^2+v^2}{R} + \frac{(1-\alpha)(1-\alpha-2u)-r^2(1+\alpha)(1+\alpha+2u)}{(1-r^2)R}] \quad .$$

Denote the right-hand side by $S(u, v)$, then we have

$$\frac{\partial S}{\partial v} = \frac{v}{2R^4}\,T(u, v) \quad ,$$

where

$$T(u, v) = 2(\alpha+u)+2R^3-(u^2+v^2)R + \frac{R}{1-r^2}\,[r^2(1+\alpha)(1+\alpha+2u) - (1-\alpha)(1-\alpha-2u)] \quad . \tag{4.3.10}$$

Now,

$$1 + \alpha = \frac{2}{1-E^2r^2} \text{ , } 1 - \alpha = -\frac{2E^2r^2}{1-E^2r^2} \ge -\frac{2r^2}{1-E^2r^2} \quad \text{as } E \le 1 \quad .$$

Hence

$$2ur^2(1+\alpha) + 2u(1-\alpha) \ge 2u(\frac{2r^2}{1-E^2r^2} - \frac{2r^2}{1-E^2r^2}) = 0 \quad ,$$

and so,

$$\begin{aligned} \frac{r^2(1+\alpha)(1+\alpha+2u)-(1-\alpha)(1-\alpha-2u)}{1 - r^2} &\ge \frac{r^2(1+\alpha)^2-(1-\alpha)^2}{1 - r^2} \\ &= \frac{1}{1-r^2}\,[\frac{4r^2}{(1-E^2r^2)^2} - \frac{4E^4r^4}{(1-E^2r^2)^2}] \\ &\ge \frac{4E^2r^2}{(1-E^2r^2)^2} = d^2 \quad . \end{aligned} \tag{4.3.11}$$

In view of (4.3.10) and (4.3.11) we get

$$T(u, v) \geq 2(\alpha+u) + 2R^3 + (d^2-u^2-v^2)R > 0 .$$

Hence the minimum of $S(u, v)$ on the disc $|p(z) - \alpha| \leq d$ is attained when $v = 0$ and $u \in [-d, d]$. Setting $v = 0$ we obtain

$$S(u, 0) = \alpha+u - \frac{1+r^2}{1-r^2} .$$

Since $dS(u, 0)/du > 0$, the minimum of $S(u, 0)$ is attained at the end-point $\alpha - d$, the value of which is

$$S(\alpha-d, 0) = -\frac{2r}{1-r^2} \cdot \frac{E+r}{1+Er} .$$

Equality in (4.3.8) occurs for the function $p_2(z)$ which is extremal for Corollary 4.3.3 .

4.3.5 <u>Remark</u>. We note that the lower bounds for $\mathrm{Re}\{p(z)\}$ and $\mathrm{Re}\{zp'(z)/p(z)\}$, $p(z) \in P[a, b]$, are attained for the same function $p_2(z)$ at $z = -r$ over the whole range $0 < r < 1$. Hence the lower bound for

$$\mathrm{Re}\{\alpha p(z) + \beta\frac{zp'(z)}{p(z)}\} , \alpha \geq 0 , \beta \geq 0 , |z| = r < 1 ,$$

over $P[a, b]$ may be obtained directly from (4.3.3) and (4.3.8) and is sharp for $p_2(z)$ at $z = -r$. This bound is useful in finding the radius of convexity of $S^*[a, b]$ (see Section 4.5) .

To investigate the univalence of the class $R[a, b]$ we need the next result.

4.3.6 <u>Theorem</u>. *If* $p(z) \in P[a, b]$, $b>1$, *then on* $|z| = r < 1$,

$$\mathrm{Re}\{p(z)+zp'(z)\} \geq \begin{cases} \dfrac{(1-Er)(1-Er-3r^2-Er^3)}{(1-r^2)(1+Er)^2}, & u_0 \leq -d , \\[2em] -\dfrac{r^4}{(1-r^2)^2}, & u_0 \geq -d , \end{cases}$$

where $u_0 = r^2/(1-r^2) - \alpha$; α, d, E *being as given in Theorem* 4.3.4 .

<u>Proof.</u> With the same argument and notation as in the proof of Theorem 4.3.4 we arrive at the inequality

$$\mathrm{Re}\{p(z)+zp'(z)\} \geq (\alpha+u)(\alpha+u-\frac{2r^2}{1-r^2}) = S(u) \quad .$$

Now, $dS/du = 0$ at $u_0 = r^2/(1-r^2) - \alpha$. Hence $S(u) \geq S(u_0)$ if $u_0 \in [-d, d]$. We have $u_0 < d$ if and only if

$$\frac{r^2}{1-r^2} < \frac{1+Er}{1-Er} \quad ,$$

that is, if and only if $1 - r^2 + r(E-r) > 0$. Returning to the expression for E we have

$$E = \frac{a + B + (1+aB)r}{1 + aB + (a+B)r} > r$$

if and only if $a + B > (a+B)r^2$, which is true for $r < 1$. Hence $E - r > 0$ and this leads to $u_0 < d$. However, the inequality $u_0 > -d$ does not always hold. Consequently, for $u_0 \geq -d$,

$$S(u) \geq S(u_0) = -\frac{r^4}{(1-r^2)^2}$$

and for $u_0 \leq -d$,

$$S(u) \geq S(-d) = \frac{(1-Er)(1-Er-3r^2-Er^3)}{(1-r^2)(1+Er)^2} \quad .$$

4.4 On the univalence and starlikeness of a class of regular functions with constraint

As mentioned previously in Section 1.3 of Chapter 1, every function $f(z)$ in the class

$$R = \{f(z) \in \mathbb{N} \ ; \ \mathrm{Re}\{f(z)/z\} > 0, \ z \in \Delta\}$$

is univalent and starlike in $|z| < \sqrt{2} - 1$ (see Polya and Szegö [66, Problem 3] and MacGregor [45]). In this section, we refine these results by considering functions in $\mathbb{R}$ with the constraint $f(a) = ab$, that is, functions of the class

$$\mathbb{R}[a, b] = \{f(z) \in \mathbb{N} \ ; \ f(z)/z \in P\,[a, b]\ , \ z \in \Delta\} \ .$$

4.4.1 Theorem. *Let* $f(z) \in \mathbb{R}[a, b]$, $b > 1$, *then on* $|z| = r$

$$\mathrm{Re}\{f'(z)\} \geq \begin{cases} \dfrac{(1-Er)(1-Er-3r^2-Er^3)}{(1-r^2)(1+Er)^2} & , \quad 0 < r \leq r_1 \ , \\[2ex] -\dfrac{r^4}{(1-r^2)^2} & , \quad r_1 \leq r < 1 \ , \end{cases}$$

where r_1 *is the only root in* $(0, 1)$ *of the equation*

$$1 + aB - 3(1+aB)r^2 - 2(a+B)r^3 = 0 \ ,$$

E, B *being as given in Theorem* 4.3.1 .

Proof. For $f(z) \in \mathbb{R}[a, b]$, the representation

$$(4.4.1) \qquad f(z) = zp(z) \quad , \ p(z) \in P[a, b] \ , \ z \in \Delta \ ,$$

implies

$$f'(z) = p(z) + zp'(z) .$$

Thus the result is readily given by Theorem 4.3.6. We note that the condition $u_0 = -d$ is equivalent to

$$1 + aB - 3(1+aB)r^2 - 2(a+B)r^3 = 0 .$$

The equation has only one positive root which is located in the interval (0, 1). Also, it is clear that the range $0 < r \leq r_1$ corresponds to $u_0 \leq -d$.

Replacing E by its value we have

$$1-Er-3r^2-Er^3 = \frac{1+aB-4(1+aB)r^2-4(a+B)r^3-(1+aB)r^4}{1+aB+(a+B)r} .$$

The numerator of the right-hand side has only one zero, r_2, in (0, 1). It may be easily checked that $r_2 < r_1$. Thus, in view of these remarks, Theorem 4.4.1 gives $\text{Re}\{f'(z)\} > 0$ in $|z| < r_2$. Consequently, an application of Wolff-Noshiro-Warschawski's theorem yields

4.4.2 <u>Corollary</u>. *Every* $f(z) \in R[a, b]$, $b>1$, *is univalent in* $|z| < r_2$, *where* r_2 *is the only root in* (0, 1) *of the equation*

$$1 + aB - 4(1+aB)r^2 - 4(a+B)r^3 - (1+aB)r^4 = 0 .$$

The function $f(z) = z[1+BT(z)+z(B+T(z))]/[1+BT(z)-z(B+T(z))]$ in $R[a, b]$ shows that the radius r_2 is best possible.

Again, let $f(z) \in \mathcal{R}[a, b]$, then from (4.4.1) we may write

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} \quad , \quad z \in \Delta \ .$$

An application of Theorem 4.3.4 now yields, on $|z| = r < 1$,

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} \geq 1 - \frac{2r}{1-r^2} \cdot \frac{E+r}{1+Er} = \frac{1-Er-3r^2-Er^3}{(1-r^2)(1+Er)} \quad .$$

The numerator has its only positive zero r_2 in (0, 1) as we have just seen above. Hence

4.4.3 Corollary. *The radius of starlikeness of* $\mathcal{R}[a, b]$, $b>1$, *is given by the only root in* (0, 1) *of the equation*

$$1 + aB - 4(1+aB)r^2 - 4(a+B)r^3 - (1+aB)r^4 = 0 \ .$$

4.5 Starlike functions with a constraint

We study in this section the class $S^*[a, b]$ of starlike functions $f(z)$ in Δ which are subject to the constraint $af'(a)/f(a) = b$, where $0 < a < 1$, $b > 1$. We shall derive the distortion theorem, the covering theorem, the radius of convexity and some coefficient bounds for $S^*[a, b]$.

As we have seen in the introduction of this chapter, the class $\Sigma^c_{(s)}$ of meromorphic convex functions with pole at a point $s \neq 0$ can be characterised explicitly in terms of functions of $P[a, b]$. This class $\Sigma^c_{(s)}$ may also be characterised by functions of the class $\Sigma^*_{(0)s}$ of meromorphic starlike functions $g(z)$ with a simple pole at the origin and satisfying $sg'(s)/g(s) = (1+s^2)/(1-s^2)$, $0 < s < 1$. In fact, let $f(z) \in \Sigma^c_{(s)}$ and define

$$g(z) = (s^2z)^{-1}(z-s)^2(1-sz)^2 f'(z) \; . \tag{4.5.1}$$

Then an easy calculation yields

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zf''(z)}{f'(z)} + \frac{z+s}{z-s} - \frac{1+sz}{1-sz} \; , \quad z \in \Delta$$

$$= -p(z) \; , \quad \text{from (4.1.1)} \; ,$$

for some $p(z) \in P[a, b]$ with $a = s$, $b = (1+s^2)/(1-s^2)$. Hence $g(z) \in \Sigma^*_{(0)s}$. Conversely, given $g(z) \in \Sigma^*_{(0)s}$, a function $f(z)$ defined by the relationship (4.5.1) will belong to $\Sigma^c_{(s)}$. Thus, in this direction, we see that certain properties pertaining to $\Sigma^c_{(s)}$ may be obtained directly from those of $\Sigma^*_{(0)s}$. We further remark that a function $g(z)$ belongs to $\Sigma^*_{(0)s}$ if and only if $1/g(z)$ belongs to $S^*[a, b]$ with $a = s$, $b = (1+s^2)/(1-s^2)$. Accordingly, we shall derive certain distortion bounds for $\Sigma^*_{(0)s}$ and $\Sigma^c_{(s)}$ from those for $S^*[a, b]$. Geometric properties for a more general class of meromorphic starlike functions with pole at the origin will be investigated in the next chapter.

We first give some distortion inequalities for $S^*[a, b]$.

4.5.1 Theorem. *Let* $f(z) \in S^*[a, b]$, $b>1$; *then on* $|z| = r < 1$,

$$\frac{(1+aB)r}{1+aB+2(a+B)r+(1+aB)r^2} \le |f(z)| \le \frac{r}{1-r^2} \cdot \left(\frac{1+r}{1-r}\right)^{(a+B)/(1+aB)} \; , \tag{4.5.2}$$

$$\frac{1+aB}{1+aB+2(a+B)r+(1+aB)r^2} \cdot \frac{1-Er}{1+Er} \le |f'(z)| \le \frac{1}{1-r^2} \cdot \left(\frac{1+r}{1-r}\right)^{(a+B)/(1+aB)} \times \frac{1+Er}{1-Er} \; , \tag{4.5.3}$$

B, E *being as given in Theorem* 4.3.1 .

Proof. For $f(z) \in S^*[a, b]$, we have

$$\log \frac{f(z)}{z} = \int_0^z \frac{p(\xi)-1}{\xi} \, d\xi \quad , \quad p(z) \in P[a, b] \; .$$

Hence taking the real part of both sides and substituting $\xi = zt$ in the integral we get

$$(4.5.4) \qquad \log\left|\frac{f(z)}{z}\right| = \int_0^1 \operatorname{Re}\left\{\frac{p(zt)-1}{t}\right\} \, dt \; .$$

Inequality (4.3.3) now yields on $|zt| = rt$

$$(4.5.5) \qquad \operatorname{Re}\left\{\frac{p(zt)-1}{t}\right\} \geq -\frac{2(a+B)r+2(1+aB)r^2t}{1+aB+2(a+B)rt+(1+aB)r^2t^2} \; .$$

Hence, from (4.5.4) and (4.5.5) ,

$$|f(z)| \geq r \exp \int_0^1 \left[-\frac{2(a+B)r+2(1+aB)r^2t}{1+aB+2(a+B)rt+(1+aB)r^2t^2}\right] dt$$

$$= \frac{(1+aB)r}{1+aB+2(a+B)r+(1+aB)r^2} \; .$$

The second inequality of (4.5.2) may be similarly derived using (4.3.4). To prove (4.5.3), we write

$$|f'(z)| = \left|\frac{f(z)}{z}\right| |p(z)| \quad , \quad p(z) \in P[a, b]$$

and apply the above results and (4.3.3) or (4.3.4) .

The lower bounds for $|f(z)|$ and $|f'(z)|$ are sharp for the function

$$(4.5.6) \qquad f(z) = z \exp \int_0^z \frac{p_2(\xi)-1}{\xi} \, d\xi \quad ,$$

where $p_2(z)$ is extremal for (4.3.3) .

Let $r \to 1$ in (4.5.2) we obtain the covering theorem for $S^*[a, b]$.

4.5.2 Corollary. *The image of the unit disc under every function in* $S^*[a, b]$, $b>1$, *contains the disc of centre* 0 *and radius* $(1+aB)/2(1+a)(1+B)$.

From the observation that $g(z) \in \sum^*_{(0)s}$ if and only if $1/g(z) \in S^*[a, b]$ with $a = s$, $b = (1+s^2)/(1-s^2)$, the following results are obtained immediately from (4.5.2).

4.5.3 Corollary. *Let* $g(z) \in \sum^*_{(0)s}$ *; then on* $|z| = r < 1$,

$$(4.5.7) \quad \left(\frac{r}{1-r^2}\right)^{-1}\left(\frac{1+r}{1-r}\right)^{-2s/(1+s^2)} \le |g(z)| \le \frac{1+s^2+4sr+(1+s^2)r^2}{(1+s^2)r} .$$

The second inequality is sharp for the function

$$g(z) = - z \exp \int_0^z \frac{p_2(\xi)+1}{\xi} \, d\xi \quad ,$$

$p_2(z)$ being as given in Theorem 4.3.1 with $a = s$, $b = (1+s^2)/(1-s^2)$.

In view of (4.5.7) and the relationship (4.5.1) we get

4.5.4 Corollary. *Let* $f(z) \in \sum^c_{(s)}$ *; then on* $|z| = r < 1$,

$$\frac{s^2(1-r^2)}{|z-s|^2|1-sz|^2}\left(\frac{1-r}{1+r}\right)^{2s/(1+s^2)} \le |f'(z)| \le \frac{s^2[1+s^2+4sr+(1+s^2)r^2]}{(1+s^2)|z-s|^2|1-sz|^2} .$$

The upper bound for $|f'(z)|$ is attained for the function

$$f(z) = \frac{s^2}{1-s^4}\left(\frac{z-s}{1-sz} - \frac{1-sz}{z-s} + s - \frac{1}{s}\right) .$$

This result was previously obtained by Pfaltzgraff and Pinchuk [64] .

Taking into account Remark 4.3.5, we prove

4.5.5 Theorem. *The radius of convexity of* $S^*[a, b]$, $b>1$, *is given by the only root in* $(0, 1)$ *of the equation*

$$1 + aB - 2(a+B)r - 6(1+aB)r^2 - 2(a+B)r^3 + (1+aB)r^4 = 0 ,$$

where B *is as given in Theorem* 4.3.1 .

Proof. For $f(z) \in S^*[a, b]$, we deduce

$$\text{(4.5.8)} \qquad \mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} = \mathrm{Re}\{p(z) + \frac{zp'(z)}{p(z)}\} ,$$

$p(z) \in P[a, b]$. Applying (4.3.3) and (4.3.8) to the terms of the right-hand side of (4.5.8) yields on $|z| = r$

$$\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \ge \frac{1-3Er-3r^2+Er^3}{(1-r^2)(1+Er)}$$

$$= \frac{1+aB-2(a+B)r-6(1+aB)r^2-2(a+B)r^3+(1+aB)r^4}{(1-r^2)(1+Er)[1+aB+(a+B)r]} .$$

It is clear that the numerator has only one zero in (0, 1). The extremal function f(z) given by (4.5.6) shows that the result is sharp.

Theorems 4.5.1 and 4.5.5 show the change in the degree of distortion of starlike functions $f(z)$ when the additional condition $af'(a)/f(a) = b$ is imposed upon $f(z)$. This influence is also apparent when we consider the coefficient bounds for these functions. We require first of all the following coefficient inequality over $P[a, b]$.

4.5.6 <u>Lemma</u>. *Let* $p(z) = 1 + p_1 z + p_2 z^2 + \dots \in P[a, b]$, $b>1$; *then*

$$(4.5.9) \qquad |p_2 - \tfrac{1}{2} p_1^2| \le 2 - \tfrac{1}{2}|p_1|^2 \quad .$$

The result is sharp.

<u>Proof</u>. From the representation formula (4.2.9) we may write

$$(4.5.10) \qquad p(z) = \frac{1+z\Phi(z)}{1-z\Phi(z)} \quad ,$$

where $\Phi(z) = [B+T(z)\psi(z)]/[1+BT(z)\psi(z)] = b_0 + b_1 z + \dots$ From Carathéodory's inequality (1.2.5) we have

$$(4.5.11) \qquad |b_1| \le 1 - |b_0|^2 \quad .$$

On substituting the series expansion for $p(z)$ and $\Phi(z)$ in (4.5.10) and equating the coefficients of powers of z, we get

$$(4.5.12) \qquad b_0 = p_1/2 \quad , \quad b_1 = (2p_1 - p_1^2)/4 \; .$$

Inequality (4.5.9) now follows from (4.5.11) and (4.5.12). We have seen in the proof of Theorem 4.3.2 that equality in (4.5.11) occurs for the function $\Phi_0(z) = [B+T(z)]/[1+BT(z)]$. Hence (4.5.9) is sharp for the function $p_2(z) = [1+z\Phi_0(z)]/[1-z\Phi_0(z)]$.

4.5.7 <u>Theorem</u>. *Let* $f(z) = z + a_2z^2 + a_3z^3 + \ldots \in S^*[a, b]$, $b>1$; *then*

$$|a_2| \le 2\,\frac{a + B}{1 + aB} \quad ,$$

$$|a_3| \le 1 + 2\left(\frac{a+B}{1+aB}\right)^2 \quad .$$

B *being as given in Theorem* 4.3.1. *The results are sharp.*

<u>Proof</u>. Equating coefficients in the expression $zf'(z)/f(z) = p(z)$, $p(z) = 1 + p_1z + p_2z^2 + \ldots \in P[a, b]$, we obtain $a_2 = p_1$, $2a_3 = p_1^2 + p_2$. The bound for $|a_2|$ is therefore given in Corollary 4.3.3. For $|a_3|$ we have

$$\begin{aligned}
2|a_3| &= |p_2 - \tfrac{1}{2} p_1^2 + \tfrac{3}{2} p_1^2| \\
&\le 2 - \tfrac{1}{2}|p_1|^2 + \tfrac{3}{2}|p_1|^2 \quad \text{, from (4.5.9)} \\
&= 2 + |p_1|^2 \\
&\le 2 + 4\left(\frac{a+B}{1+aB}\right)^2 \quad \text{, from (4.3.6)} \quad .
\end{aligned}$$

Hence $|a_3| \le 1 + 2[(a+B)/(1+aB)]^2$. Since (4.5.9) and (4.3.6) are sharp for the function $p_2(z)$, the bounds for $|a_2|$ and $|a_3|$ are attained for the function $f(z)$ given by (4.5.6) .

CHAPTER 5

MEROMORPHIC STARLIKE FUNCTIONS

5.1 Introduction

In this chapter we consider functions which are meromorphic and univalent in the unit disc. The univalence of these functions requires that they can have no other singularities but a simple pole. There is no loss of generality when the univalent functions with a simple pole in Δ are normalised so that the pole is located at the origin and the residue of the pole has the value 1. The class of these functions, that is, functions which are univalent in Δ with Laurent expansion

$$(5.1.1) \qquad f(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + \dots ,$$

will be denoted by $\sum$.

There has been considerable interest in the subclass $\sum^*$ of functions in $\sum$ which map the unit disc onto domains whose complements are starlike with respect to the origin. These functions are characterised by the condition

$$(5.1.2) \qquad \mathrm{Re}\,\{-\frac{zf'(z)}{f(z)}\} > 0 \quad , z \in \Delta \quad .$$

The problem on coefficient bounds for $\sum^*$ was partially solved by Nehari and Netanyahu [60] and completely settled by Clunie [17]. Clunie's method has become standard in dealing with coefficient problems for various subclasses of S^* and $\sum^*$ alike (see, for example, Libera and Livingston [41]). Variational formulae were developed by Royster [78] to tackle extremal problems over $\sum^*$.

In recent years, interesting subclasses of $\sum^*$ have been introduced and studied; for instance,

$$\textstyle\sum_\alpha^* = \{f(z) = \frac{1}{z} + a_0 + a_1 z + \ldots ; \ \mathrm{Re}\{-\frac{zf'(z)}{f(z)}\} > \alpha, \ 0 \le \alpha < 1 \ , \ z \in \Delta\},$$

$$\textstyle\sum^*[\alpha] = \{f(z) = \frac{1}{z} + a_0 + a_1 z + \ldots; \ |(\frac{zf'(z)}{f(z)} + 1)/(\frac{zf'(z)}{f(z)} - 1)| < \alpha \ , \quad 0 < \alpha \le 1 \ , \ z \in \Delta\} \ ,$$

$$\textstyle\sum^*(M) = \{f(z) = \frac{1}{z} + a_0 + a_1 z + \ldots ; \ |\frac{zf'(z)}{f(z)} + M| < M \ , \ M > \frac{1}{2} \ , \ z \in \Delta\} \ .$$

Robertson [76] posed the question on the radius of convexity of $\sum_\alpha^*$ and solved the case $\alpha = \frac{1}{2}$. This problem was settled by Zmorovič [100] and Singh and Goel [85] . The class $\sum^*[\alpha]$ was introduced by Padmanabhan [62] who found its radius of convexity. Wiatrowski [95] derived distortion bounds and the radius of convexity for $\sum^*(M)$. Hence, to a certain extent, the work on subclasses of starlike functions in the meromorphic case has paralleled that in the regular case.

Quite recently, Karunakaran [35] considered the class $\sum^*(A, B)$ of functions in $\sum$ which are defined by

$$-\frac{zf'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad , \ w(z) \in \mathcal{B}, \quad z \in \Delta \quad , \tag{5.1.3}$$

where A, B are restricted by the conditions $-1 \le B \le 0$, $B < A \le -B$. These conditions are not general enough to cover such cases as $\sum^*(M) \equiv \sum^*(1, 1/M - 1)$ in which $B > 0$ for $\frac{1}{2} < M < 1$ and $\sum^*_{(\alpha)} \equiv \sum^*(\alpha, 0)$, which we define below.

In this chapter we shall investigate the class $\sum^*(A, B)$ with

A, B subject to the general conditions that $-1 \le B < A \le 1$. Our results will extend the corresponding ones due to Karunakaran [35].

As seen previously for the regular case, the class $\Sigma^*(A, B)$ reduces to certain subclasses of Σ^* by appropriate choices of A and B as follows.

$$\Sigma^*(1-2\alpha, -1) \equiv \Sigma^*_\alpha , \Sigma^*(\alpha, -\alpha) \equiv \Sigma^*[\alpha], \Sigma^*(1, 1/M-1) \equiv \Sigma^*(M) ,$$

$$\Sigma^*(\alpha, 0) \equiv \Sigma^*_{(\alpha)} = \{f(z) = \frac{1}{z} + a_0 + a_1 z + \ldots; \left|\frac{zf'(z)}{f(z)} + 1\right| < \alpha ,$$

$$0 < \alpha \le 1 , z \in \Delta\} .$$

Wiatrowski [95] obtained sharp bounds for $|f'(z)|$, $f(z) \in \Sigma^*(M)$. We derive the corresponding results for $\Sigma^*[\alpha]$. The lower bound and partial result for the upper bound for $|f'(z)|$ over Σ^*_α will also be given. The problem of determining sharp bounds for $|f'(z)|$ over $\Sigma^*(A, B)$ remains open.

Part of this chapter will be devoted to a study of another subclass of Σ^* , namely, the class

$$\Sigma^*(\alpha) = \{f(z)=\frac{1}{z}+a_0+a_1z+\ldots; (1-\frac{\alpha}{2})\pi<\arg\{\frac{zf'(z)}{f(z)}\}<(1+\frac{\alpha}{2})\pi , 0<\alpha\le 1, z\in\Delta\}$$

introduced by Brannan, Clunie and Kirwan [12]. This is the parallel of the subclass

$$S^*(\alpha) = \{f(z) \in N ; \left|\arg\{\frac{zf'(z)}{f(z)}\}\right| < \alpha\frac{\pi}{2} , 0 < \alpha \le 1, z \in \Delta\}$$

investigated by Brannan and Kirwan [11] and Stankiewicz [86] . Brannan, Clunie and Kirwan [12] obtained the coefficient bounds for

$\sum^*(\alpha)$. We shall derive the distortion theorem, the covering theorem and the radius of convexity for $\sum^*(\alpha)$.

The problems on the radius of convexity of $\sum^*(A, B)$ and $\sum^*(\alpha)$ may be reduced to the extremal problems

$$(5.1.4) \qquad \min_{p(z)\in P(A,\, B)} \ \min_{|z|=r<1} \ \mathrm{Re}\{p(z) - \frac{zp'(z)}{p(z)}\}$$

and

$$(5.1.5) \qquad \min_{p(z)\in P} \ \min_{|z|=r<1} \ \mathrm{Re}\{p(z)^{\alpha} - \alpha\frac{zp'(z)}{p(z)}\} \ , \ 0 < \alpha \le 1 \ .$$

These are solved making use of Dieudonné's lemma. However, problem (5.1.5) requires a different argument as here, we have a term in power of $p(z)$.

5.2 The functional $\mathrm{Re}\{\gamma p(z) - zp'(z)/p(z)\}$, $\gamma \le 1$, over $P(A, B)$

5.2.1 Theorem. *If* $p(z) \in P(A, B)$, $-(1+B)/(A-B) \le \gamma \le 1$, *then on* $|z| = r < 1$,

$$\mathrm{Re}\{\gamma p(z) - \frac{zp'(z)}{p(z)}\} \ge \begin{cases} \dfrac{\gamma-[(1-2\gamma)A-B]r+\gamma A^2r^2}{(1+Ar)(1+Br)} \ , & R_5 \ge R_6 \ , \\[2ex] -\dfrac{A+B}{A-B} + \dfrac{2}{(A-B)(1-r^2)}[(L_2K_2)^{\frac{1}{2}}-(1-ABr^2)], & R_6 \ge R_5, \end{cases}$$

where $R_5 = (L_2/K_2)^{\frac{1}{2}}$, $R_6 = (1+Ar)/(1+Br)$, $L_2 = (1+A)(1-Ar^2)$, $K_2 = \gamma(A-B)(1-r^2) + (1+B)(1-Br^2)$. *The result is sharp.*

Proof. With the same argument as in the proof of Theorem 1.2.2 using (0.1.7) and (1.2.4) we get

$$(5.2.1)\quad \mathrm{Re}\{\gamma p(z) - \frac{zp'(z)}{p(z)}\} \geq -\frac{A+B}{A-B} + \frac{1}{A-B}\mathrm{Re}\{[\gamma(A-B)+B]p(z) + \frac{A}{p(z)}\}$$

$$- \frac{r^2|Bp(z)-A|^2 - |1-p(z)|^2}{(A-B)(1-r^2)|p(z)|} .$$

Put $p(z) = Re^{i\theta}$, where $R \in [a - d, a + d]$ with a, d being given by (1.2.2) and denote the right-hand side of (5.2.1) by $S(R, \theta)$, then

$$(5.2.2)\quad S(R, \theta) = -\frac{A+B}{A-B} + \frac{1}{A-B}\{[(\gamma(A-B)+B)R+\frac{A}{R} - \frac{2(1-ABr^2)}{1-r^2}] \cos\theta$$

$$+ \frac{1-A^2r^2}{1-r^2} . \frac{1}{R} + \frac{1-B^2r^2}{1-r^2} . R\} .$$

Now,

$$\frac{\partial S}{\partial \theta} = \frac{\sin\theta}{A-B} T(R) ,$$

where

$$T(R) = 2\,\frac{1-ABr^2}{1 - r^2} - \frac{A}{R} - [\gamma(A-B)+B]R$$

$$\geq 2\,\frac{1-ABr^2}{1 - r^2} - (\frac{1}{R} + R) \quad \text{as} \quad A \leq 1 \text{ and } \gamma \leq 1.$$

Denote the right-hand side by $F(R)$; then $dF/dR = 1/R^2 - 1$. Since $R \in [a - d, a + d]$ and $a - d < 1$, $a + d > 1$, the minimum of $F(R)$ is attained at either $R = a - d$ or $R = a + d$. Now,

$$F(a-d) = 2\,\frac{1-ABr^2}{1 - r^2} - \frac{1-Br}{1-Ar} - \frac{1-Ar}{1-Br}$$

$$= \frac{r^2[(1-A^2)(1-Br)^2+(1-B^2)(1-Ar)^2]}{(1-r^2)(1-Ar)(1-Br)} > 0 .$$

Also,

$$F(a+d) = 2\frac{1-ABr^2}{1-r^2} - \frac{1+Br}{1+Ar} - \frac{1+Ar}{1+Br}$$

$$= \frac{r^2[(1-A^2)(1+Br)^2 + (1-B^2)(1+Ar)^2]}{(1-r^2)(1+Ar)(1+Br)} > 0 .$$

Thus $T(R) > 0$. And so, the minimum of $S(R,\theta)$ on the disc $|p(z)-a| \le d$ is attained when $\theta = 0$ and $R \in [a-d,a+d]$. Setting $\theta = 0$ in (5.2.2) we get

$$S(R,0) = -\frac{A+B}{A-B} + \frac{1}{A-B}\{[\gamma(A-B)+B+\frac{1-B^2r^2}{1-r^2}]R + (A+\frac{1-A^2r^2}{1-r^2})\frac{1}{R} - 2\frac{1-ABr^2}{1-r^2}\}$$

which yields

$$\frac{dS(R,0)}{dR} = \frac{1}{A-B}[\gamma(A-B) + B + \frac{1-B^2r^2}{1-r^2} - \frac{(1+A)(1-Ar^2)}{1-r^2} \cdot \frac{1}{R^2}] .$$

In the above expression we have that

$$\gamma(A-B) + B + \frac{1-B^2r^2}{1-r^2} \ge \gamma(A-B) + B + 1 \ge 0$$

if $\gamma \ge -(1+B)/(A-B)$. Thus for $-(1+B)/(A-B) \le \gamma \le 1$, the minimum of $S(R,0)$ occurs at $R = R_5$ if $R_5 \in [a-d,a+d]$, its value being

$$S(R_5,0) = -\frac{A+B}{A-B} + \frac{2}{(A-B)(1-r^2)}[(L_2K_2)^{\frac{1}{2}} - (1-ABr^2)] .$$

We next want to show that $R_5 > a-d$. Indeed, for γ in the range $-(1+B)/(A-B) \le \gamma \le 1$, we have

$$\frac{(1+A)(1-Ar^2)}{\gamma(A-B)(1-r^2)+(1+B)(1-Br^2)} > \frac{1-Ar^2}{1-Br^2}$$

if and only if $1-Br^2 > \gamma(1-r^2)$, that is, if and only if $1 > (B-\gamma)r^2/(1-\gamma)$, which is always true as $(B-\gamma)/(1-\gamma) < 1$ for $\gamma \le 1$. Consequently,

$$R_5^2 > \frac{1-Ar^2}{1-Br^2} > \frac{1-Ar}{1-Br} > \left(\frac{1-Ar}{1-Br}\right)^2 = (a-d)^2 \quad .$$

In other words, $R_5 > a-d$. However, R_5 is not always less than $a+d$. For $R_5 \ge a+d = R_6$, the minimum of $S(R,0)$ occurs at $R = R_6$, its value being

$$S(R_6,0) = \frac{\gamma - [(1-2\gamma)A-B]r + \gamma A^2r^2}{(1+Ar)(1+Br)} \quad .$$

The result is sharp for the function $p_1(z) = (1+Az)/(1+Bz)$ for $R_5 \ge R_6$ and the function $p_3(z) = [1+Aw_2(z)]/[1+Bw_2(z)]$ for $R_6 \ge R_5$, where $w_2(z) = z(z-c_2)/(1-c_2z)$ with c_2 being determined from the equation $\mathrm{Re}\{[1+Aw_2(z)]/[1+Bw_2(z)]\} = R_5$ at $z = r$.

In the above theorem, we have restricted γ so that $\gamma \ge -(1+B)/(A-B)$. In the next theorem, which gives the upper bound for $\mathrm{Re}\{\gamma p(z)-zp'(z)/p(z)\}$, we shall see that no lower limit for the values of γ is required.

5.2.2 <u>Theorem</u>. *If* $p(z) \in P(A, B)$, $\gamma \le 1$, *then on* $|z| = r < 1$

$$\mathrm{Re}\{\gamma p(z)- \frac{zp'(z)}{p(z)}\} \le \begin{cases} \dfrac{\gamma + [(1-2\gamma)A-B]r + \gamma A^2r^2}{(1-Ar)(1-Br)} , & R_7 \le R_2 , \\[2ex] -\dfrac{A+B}{A-B} + \dfrac{2}{(A-B)(1-r^2)}[1-ABr^2-(L_3K_3)^{\frac{1}{2}}] , & R_2 \le R_7 , \end{cases}$$

where $R_7 = (L_3/K_3)^{\frac{1}{2}}$, $R_2 = (1-Ar)/(1-Br)$, $L_3 = (1-A)(1+Ar^2)$, $K_3 = (1-B)(1+Br^2) - \gamma(A-B)(1-r^2)$. *The result is sharp.*

Proof. The same argument as in the proof of Theorem 1.2.2 yields

$$(5.2.3)\quad \mathrm{Re}\{\gamma p(z)-\frac{zp'(z)}{p(z)}\} \le -\frac{A+B}{A-B}+\frac{1}{A-B}\mathrm{Re}\{[\gamma(A-B)+B]p(z)+\frac{A}{p(z)}\}$$

$$+\frac{r^2|Bp(z)-A|^2-|1-p(z)|^2}{(A-B)(1-r^2)|p(z)|}\quad .$$

Put $p(z) = a+u+iv$ and denote the right-hand side of (5.2.3) by $S(u,v)$ then

$$(5.2.4)\quad S(u,v) = -\frac{A+B}{A-B}+\frac{1}{A-B}\{[\gamma(A-B)+B](a+u)+\frac{A(a+u)}{R^2}+\frac{1-B^2r^2}{1-r^2}\cdot\frac{d^2-u^2-v^2}{R}\},$$

so that

$$\frac{\partial S}{\partial v} = -\frac{1}{A-B}\cdot\frac{v}{R^4}T(u,v)\quad ,$$

where

$$T(u,v) = 2A(a+u)+\frac{1-B^2r^2}{1-r^2}[2R^3+(d^2-u^2-v^2)R]$$

$$\ge 2(a+u)[A+\frac{1-B^2r^2}{1-r^2}(a-d)^2] > 0 \quad \text{from (1.2.9).}$$

Hence the maximum of $S(u,v)$ on the disc $|p(z)-a| \le d$ is attained when $v = 0$ and $u \in [-d,d]$. Putting $v = 0$ in (5.2.4) gives

$$S(u,0) = -\frac{A+B}{A-B}+\frac{1}{A-B}\{(A-\frac{1-A^2r^2}{1-r^2})\frac{1}{a+u}+[\gamma(A-B)+B-\frac{1-B^2r^2}{1-r^2}](a+u)+2\frac{1-ABr^2}{1-r^2}\}$$

which yields

$$\frac{dS(u,0)}{du} = \frac{1}{(A-B)(1-r^2)}[\gamma(A-B)(1-r^2)-(1-B)(1+Br^2)+(1-A)(1+Ar^2)\frac{1}{(a+u)^2}]\ .$$

Now $(1-B)(1+Br^2)-\gamma(A-B)(1-r^2) > 0$ if and only if

$$\gamma < \frac{1-B}{A-B} \cdot \frac{1+Br^2}{1-r^2} \quad .$$

Since $1-B \geq A-B$ and $(1+Br^2)/(1-r^2) \geq 1$, the restriction $\gamma \leq 1$ shows that the above condition is satisfied. Hence with $\gamma \leq 1$, we see that $dS(u,0)/du$ vanishes at $u_0 = (L_3/K_3)^{\frac{1}{2}} - a$. Thus the maximum of $S(u,0)$ occurs at $u = u_0$ if $u_0 \in [-d,d]$, its value being

$$S(u_0,0) = -\frac{A+B}{A-B} + \frac{2}{(A-B)(1-r^2)}[1-ABr^2-(L_3K_3)^{\frac{1}{2}}] \quad .$$

Now, an easy calculation shows that

$$\frac{(1-A)(1+Ar^2)}{(1-B)(1+Br^2)-\gamma(A-B)(1-r^2)} < \frac{1+Ar^2}{1+Br^2}$$

if and only if $\gamma(1-r^2) < 1+Br^2$, which holds for $\gamma \leq 1$. Hence from (1.2.10)

$$(a+u_0)^2 < \frac{1+Ar^2}{1+Br^2} < (a+d)^2 \quad ,$$

that is, $u_0 < d$. However, it is not necessary that $u_0 > -d$. For $u_0 \leq -d$, that is, $R_7 \leq R_2$, the maximum of $S(u,0)$ occurs at $u_0 = -d$, its value being

$$S(-d,0) = \frac{\gamma + [(1-2\gamma)A-B]r + \gamma A^2r^2}{(1-Ar)(1-Br)} \quad .$$

The result is sharp for the function $p_1(z) = (1+Az)/(1+Bz)$ for $R_7 \leq R_2$ and the function $p_4(z) = [1+Aw_3(z)]/[1+Bw_3(z)]$ for $R_2 \leq R_7$, where $w_3(z) = z(z-c_3)/(1-c_3z)$ with c_3 such that $\mathrm{Re}\{[1+Aw_3(z)]/[1+Bw_3(z)]\} = R_7$ at $z = -r$.

5.3 The class $\sum^*(A, B)$ of meromorphic starlike functions

We recall that

$$\sum^*(A, B) = \{f(z) = 1/z + a_0 + a_1 z + \dots \; ; \; -\frac{zf'(z)}{f(z)} \in P(A, B) \; , \; z \in \Delta\} \; .$$

This section establishes the radius of convexity and the bounds for $|f(z)|$ for $\sum^*(A, B)$. As mentioned previously, the bounds for $|f'(z)|$ over $\sum^*(A, B)$ are not known. However, we shall determine these bounds for several special cases of $\sum^*(A, B)$. In particular, for the class $\sum_\alpha^*$, the bounds obtained improve upon certain results due to Pommerenke [67] . The coefficient bounds for $\sum^*(A, B)$ were established by Libera and Livingston [41] .

5.3.1 Theorem. *The radius of convexity of* $\sum^*(A, B)$ *is given by the smallest root in* $(0,1]$ *of*

(i) $A^2r^2 + (A+B)r + 1 = 0$, *for* $R_6 \le R_5$;

(ii) $(4A^2+3A+B)r^4 - 2[2(1+A)^2+A-B]r^2 + 4+3A+B = 0$, *for* $R_5 \le R_6$,

R_5 , R_6 *being as given in Theorem* 5.2.1 *with* $\gamma = 1$.

Proof. For $f(z) \in \sum^*(A, B)$, we deduce

$$(5.3.1) \qquad -[1+\frac{zf''(z)}{f'(z)}] = p(z) - \frac{zp'(z)}{p(z)} \; , \quad p(z) \in P(A, B) \; .$$

The result now follows from Theorem 5.2.1 with $\gamma = 1$ and is sharp for the functions $f_1(z)$ for $R_6 \le R_5$ and $f_3(z)$ for $R_5 \le R_6$, where $f_1(z)$, $f_3(z)$ are given by

$$-\frac{zf_i'(z)}{f_i(z)} = p_i(z) \quad , \quad i = 1,3 \; ,$$

$p_1(z)$, $p_3(z)$ being extremal for Theorem 5.2.1.

As a special case of Theorem 5.3.1, we determine the radius of convexity of the class $\Sigma^*_{(\alpha)}$ of functions $f(z) = 1/z+a_0+a_1z+\ldots$ for which

$$\left|\frac{zf'(z)}{f(z)}+1\right| < \alpha \ , \ 0 < \alpha \le 1 \ , \ z \in \Delta \ .$$

Its parallel in the regular case, that is, the class $S^*_{(\alpha)}$ of functions $f(z) \in N$ which satisfy

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \alpha \ , \ 0 < \alpha \le 1, \ z \in \Delta \ ,$$

was thoroughly investigated by Wright [97] , Bajpai [7] , Eenigenburg [19] and McCarty [52] .

5.3.2 <u>Corollary</u>. *The radius of convexity of* $\Sigma^*_{(\alpha)}$ *is*

$$\sigma = \{[2+5\alpha+2\alpha^2-2(1+\alpha)(1+\alpha^2)^{\frac{1}{2}}]/\alpha(4\alpha+3)\}^{\frac{1}{2}} \ .$$

<u>Proof</u>. For $f(z) \in \Sigma^*_{(\alpha)}$ we may write

$$-\frac{zf'(z)}{f(z)} = p(z) \ , \quad z \in \Delta \ ,$$

where $p(z)$ satisfies the condition

$$|p(z) - 1| < \alpha \ , \ 0 < \alpha \le 1 \ , \ z \in \Delta \ .$$

Put $w(z) = [p(z)-1]/\alpha$; then $w(z) \in B$ and $p(z) = 1+\alpha w(z)$. Hence $p(z) \in P(\alpha,0)$. Theorem 5.3.1 with $A = \alpha$, $B = 0$ gives, for $R_5 \le R_6$, the radius of convexity of $f(z)$ to be the smallest root in $(0,1]$ of the equation

$$\alpha(4\alpha+3)r^4 - 2(2+5\alpha+2\alpha^2)r^2 + 4+3\alpha = 0 \ .$$

It is clear that the only root in (0,1) of this equation is σ. Now, the condition $R_5 \le R_6$ with $A = \alpha$, $B = 0$, $\gamma = 1$ is equivalent to

$$-2(1+\alpha) - \alpha(2+\alpha)r + 2\alpha r^2 + \alpha^2 r^3 \le 0 \quad ,$$

which always holds for $0 < \alpha \le 1$, $0 < r < 1$. Hence the case $R_6 \le R_5$ does not exist for the class $\sum^*_{(\alpha)}$. The proof is therefore completed.

To obtain bounds for $|f(z)|$ over $\sum^*(A, B)$, we observe that $f(z) \in \sum^*(A, B)$ if and only if $1/f(z) \in S^*(A, B)$. Hence an application of Theorem 2.3.1 with $k = 1$ gives

5.3.3 <u>Corollary</u>. *Let* $f(z) \in \sum^*(A, B)$; *then on* $|z| = r < 1$,

$$r^{-1}(1+Br)^{(B-A)/B} \le |f(z)| \le r^{-1}(1-Br)^{(B-A)/B}, \quad \textit{if} \quad B \ne 0 ,$$

$$r^{-1}\exp(-Ar) \le |f(z)| \le r^{-1}\exp(Ar) \quad , \quad \textit{if} \quad B = 0 .$$

The function $f(z)$ defined by

$$-\frac{zf'(z)}{f(z)} = \frac{1+Az}{1+Bz} \quad , \quad z \in \Delta$$

shows that the bounds are sharp.

We next derive bounds for $|f'(z)|$ for two subclasses of $\sum^*$, namely, $\sum^*_\alpha \equiv \sum^*(1-2\alpha,-1)$ and $\sum^*[\alpha] \equiv \sum^*(\alpha,-\alpha)$.

The class $\sum^*_\alpha$ was introduced by Pommerenke [67] and the class $\sum^*[\alpha]$ by Padmanabhan [62] . In fact, Pommerenke [67] considered the class $\P(\alpha)$ of functions $f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$ which are regular in $1 < |z| < \infty$ and satisfy there

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} > \alpha \ , \ 0 \le \alpha < 1 \ .$$

For these functions, he proved

(i) as $r \to 1$,

$$\max_{f(z)\in\P(\alpha)} \max_{|z|\le r}|f'(z)| \sim \frac{2^{2(1-\alpha)}e^{-1}}{(1-r^{-1})\log 1/(1-r^{-1})} ,$$

(ii) $|f'(z)| \ge \alpha\left|\frac{f(z)}{z}\right| + (1-\alpha)\left|\frac{z}{f(z)}\right|^{\alpha/(1-\alpha)}(1-\frac{1}{|z|^2})$

$$\ge (1-\frac{1}{|z|^2})^{1-\alpha} .$$

Equality is attained in (ii) for the function

$$f(z) = z(1-\frac{2}{rz}+\frac{1}{z^2})^{1-\alpha}$$

for $|z| = r > 1$.

We observe that this function depends on r. Therefore it is extremal only on the circle $|z| = r$ and not uniformly throughout the region $1 < |z| < \infty$.

Since the functions in $\sum_\alpha^*$ correspond to those in $\P(\alpha)$ by the transformation $z \leftrightarrow 1/z$, the above results of Pommerenke can be stated for functions in $\sum_\alpha^*$ as follows.

(i) As $r \to 1$,

$$\max_{f(z)\in\sum_\alpha^*} \max_{|z|\le r} |f'(z)| \sim \frac{2^{2(1-\alpha)}e^{-1}}{r^2(1-r)\log 1/(1-r)} ,$$

(ii) $|f'(z)| \ge \frac{1}{|z|^2}(1-|z|^2)^{1-\alpha}$, $f(z) \in \sum_\alpha^*$.

In the following we shall give the sharp upper bound for $|f'(z)|$ for $0 < r < 1$ and $0 \le \alpha \le \alpha_0$, where α_0 lies in the interval $[4/5,1)$.

The sharp lower bound for $|f'(z)|$ will also be derived for $0 < r < 1$ and for all α in $[0,1)$. This bound is reached by a function which is extremal uniformly in the disc $0 < |z| < 1$.

5.3.4 Theorem. *Let* $f(z) \in \sum_{\alpha}^{*}$, $\beta = 1-2\alpha$; *then on* $|z| = r < 1$

$$|f'(z)| \le \begin{cases} r^{-2}(1-r^2)^{-1} , & \alpha = 0 , \\ \dfrac{1}{4r^2}[1+(\dfrac{1-\beta r^2}{1-r^2})^{\frac12}]^2\{\dfrac{(1+\sqrt{\beta})[(1-\beta r^2)^{\frac12}-\sqrt{\beta}(1-r^2)^{\frac12}]}{(1-\sqrt{\beta})[(1-\beta r^2)^{\frac12}+\sqrt{\beta}(1-r^2)^{\frac12}]}\}^{\sqrt{\beta}} , & 0 < \alpha \le \frac12 , \\ \dfrac{1}{r^2}[1+(\dfrac{1-\beta r^2}{1-r^2})^{\frac12}]^2 \exp\{2\sqrt{-\beta}\tan^{-1}\dfrac{\sqrt{-\beta}[1-(1-r^2)^{\frac12}(1-\beta r^2)^{\frac12}]}{1+\beta-\beta r^2}\} , & \frac12 \le \alpha \le \alpha_0 , \end{cases}$$

where $4/5 \le \alpha_0 < 1$;

$$|f'(z)| \ge \begin{cases} 1/r^2 - 1 , & \alpha = 0 , \\ r^{-2}(1-r^2)^{\beta/(1-\alpha)}\{\dfrac{1+[(1+\beta r^2)/(1-r^2)]^{\frac12}}{2}\}^{-2\alpha/(1-\alpha)} & \\ \times \exp\{\dfrac{2\alpha\sqrt{\beta}}{1-\alpha}\tan^{-1}\dfrac{\sqrt{\beta}[1-(1-r^2)^{\frac12}(1+\beta r^2)^{\frac12}]}{2\alpha+\beta r^2}\} , & 0 < \alpha \le \frac12 , \\ r^{-2}(1-r^2)^{\beta/(1-\alpha)}\{\dfrac{1+[(1+\beta r^2)/(1-r^2)]^{\frac12}}{2}\}^{-2\alpha/(1-\alpha)} & \\ \times \{\dfrac{(1-\sqrt{-\beta})[(1+\beta r^2)^{\frac12}+\sqrt{-\beta}(1-r^2)^{\frac12}]}{(1+\sqrt{-\beta})[(1+\beta r^2)^{\frac12}-\sqrt{-\beta}(1-r^2)^{\frac12}]}\}^{\alpha\sqrt{-\beta}/(1-\alpha)} , & \frac12 \le \alpha < 1 . \end{cases}$$

The results are sharp.

Proof. From the expression

$$\log(z^2f'(z)) = \log|z^2f'(z)| + i\arg\{z^2f'(z)\} ,$$

we derive

$$2 + \mathrm{Re}\{\frac{zf''(z)}{f'(z)}\} = r\frac{\partial}{\partial r}\log |z^2f'(z)| \quad .$$

This together with (5.3.1) give, for $f(z) \in \sum_\alpha^*$,

$$(5.3.2) \qquad r\frac{\partial}{\partial r}\log |z^2f'(z)| = 1-\mathrm{Re}\{p(z)- \frac{zp'(z)}{p(z)}\} \quad , \quad p(z) \in P_\alpha \quad .$$

The condition $R_5 \le R_6$ of Theorem 5.2.1 with $A = 1-2\alpha$, $B = -1$, $\gamma = 1$ is equivalent to

$$(5.3.3) \qquad F(\alpha) \equiv -2r(1+r)\alpha^2+(r^2+5r+2)\alpha-2(1+r) \le 0 \quad .$$

Now, $F(0) = -2(1+r) < 0$, $F(1) = r(1-r) > 0$, $F(4/5) = -2(6r^2-9r+5)/25 < 0$ for $0 < r < 1$. Hence $F(\alpha)$ has a zero in $[4/5,1)$. It may be checked that this is the only zero, denoted by α_0, less than 1 of $F(\alpha)$. Thus for $\alpha \le \alpha_0$, we have $F(\alpha) \le 0$ for $0 < r < 1$. And so the case $R_6 \le R_5$ does not exist for $0 \le \alpha \le \alpha_0$ when we consider the class $\sum_\alpha^*$. Theorem 5.2.1 with $A = 1-2\alpha$, $B = -1$, $\gamma = 1$ applied to (5.3.2) yields

$$r\frac{\partial}{\partial r}\log |z^2f'(z)| \le 2\{\frac{1-[(1-\beta r^2)(1-r^2)]^{\frac{1}{2}}}{1-r^2}\} \quad .$$

Hence

$$\log |z^2f'(z)| \le 2\int_0^r \frac{1-[(1-\beta t^2)(1-t^2)]^{\frac{1}{2}}}{t(1-t^2)}\,dt$$

$$= 2\int_0^r \frac{t[2(1-\alpha)-\beta t^2]dt}{(1-t^2)\{1+(1-t^2)[(1-\beta t^2)/(1-t^2)]^{\frac{1}{2}}\}} \quad .$$

With the substitution $u = [(1-\beta t^2)/(1-t^2)]^{\frac{1}{2}}$, the integration may be carried out to give the upper bound for $|f'(z)|$. To obtain the lower bound for $|f'(z)|$, we note first of all that the condition $R_2 \le R_7$ of Theorem 5.2.2 with $A = 1-2\alpha$, $B = -1$, $\gamma = 1$ is equivalent to the inequality

$$2 + (1+2\alpha)r + (1-2\alpha)r^3 \geq 0 \quad ,$$

which always holds for $0 < r < 1$, $0 \leq \alpha < 1$. Hence there is only one case, $R_2 \leq R_7$, for the upper bound of $\mathrm{Re}\{p(z)-zp'(z)/p(z)\}$ with $p(z) \in P_\alpha$. This result applied to (5.3.2) gives

$$r\frac{\partial}{\partial r}\log |z^2f'(z)| \geq \frac{\beta}{1-\alpha} - \frac{1}{(1-\alpha)(1-r^2)}\{1+\beta r^2-2\alpha[(1+\beta r^2)(1-r^2)]^{\frac{1}{2}}\} \quad .$$

Hence

$$\log |z^2f'(z)| \geq \frac{1}{1-\alpha}\int_0^r\{\frac{-2\beta t}{1-t^2} -2\alpha\frac{1-[(1+\beta t^2)(1-t^2)]^{\frac{1}{2}}}{t(1-t^2)}\}dt$$

$$(5.3.4) \qquad = \frac{\beta}{1-\alpha}\log(1-r^2)- \frac{2\alpha}{1-\alpha}\int_0^r \frac{t(2\alpha+\beta t^2)dt}{(1-t^2)\{1+(1-t^2)[(1+\beta t^2)/(1-t^2)]^{\frac{1}{2}}\}} \quad .$$

It follows at once from (5.3.4) that, for $\alpha = 0$, $|f'(z)| \geq 1/r^2-1$. For $0 < \alpha < 1$, with the substitution $u = [(1+\beta t^2)/(1-t^2)]^{\frac{1}{2}}$ and carrying out the integration, we get the lower bound for $|f'(z)|$.

The upper bound for $|f'(z)|$ is attained for the function $f(z)$ defined by

$$-\frac{zf'(z)}{f(z)} = p_3(z)$$

while the lower bound for $|f'(z)|$ occurs for the function $f(z)$ defined by

$$-\frac{zf'(z)}{f(z)} = p_4(z) \quad ,$$

where $p_3(z)$, $p_4(z)$ are extremal for Theorems 5.2.1 and 5.2.2 respectively.

Padmanabhan [62] in his work on $S^*[\alpha]$ and $\sum^*[\alpha]$ derived the radius of convexity of $\sum^*[\alpha]$, while the distortion theorem for this class was not given. Here we prove

5.3.5 Theorem. *If* $f(z) \in \sum^{*}[\alpha]$, *then on* $|z| = r < 1$

$$\frac{(1-r^2)^\alpha}{r^2} \le |f'(z)| \le \frac{1}{r^2(1-r^2)^\alpha} \quad .$$

The results are sharp.

Proof. Denote by $P[\alpha]$ the class of functions $p(z) = 1+p_1z+\ldots$ which satisfy the condition

$$\left|\frac{p(z)-1}{p(z)+1}\right| < \alpha \ , \ 0 < \alpha \le 1 \ , \ z \in \Delta \quad ,$$

that is, $P[\alpha] \equiv P(\alpha,-\alpha)$. For $f(z) \in \sum^{*}[\alpha]$, we may write

$$(5.3.5) \qquad r\frac{\partial}{\partial r}\log |z^2f'(z)| = 1 - \mathrm{Re}\{p(z)- \frac{zp'(z)}{p(z)}\}$$

as in the proof of Theorem 5.3.4 , where now $p(z) \in P[\alpha]$. The condition $R_5 \le R_6$ of Theorem 5.2.1 with $A = \alpha$, $B = -\alpha$, $\gamma = 1$ is equivalent to $-2(1+\alpha)(1-\alpha r^2) \le 0$, which is always true for $0 < r < 1$, $0 < \alpha \le 1$. Hence the case $R_6 \le R_5$ does not exist for $p(z) \in P[\alpha]$. Consequently, an application of Theorem 5.2.1 to (5.3.5) yields

$$r\frac{\partial}{\partial r}\log |z^2f'(z)| \le \frac{2\alpha r^2}{1-r^2} \quad .$$

And so,

$$\log |z^2f'(z)| \le \int_0^r \frac{2\alpha t dt}{1-t^2} = -\alpha\log(1-r^2) \quad ,$$

that is, $|f'(z)| \le r^{-2}(1-r^2)^{-\alpha}$. Similarly, we can show that the case $R_7 \le R_2$ of Theorem 5.2.2 does not exist for $p(z) \in P[\alpha]$ and the lower bound for $|f'(z)|$ can be derived from Theorem 5.2.2 with $A = \alpha$, $B = -\alpha$, $\gamma = 1$ and (5.3.5) .

The upper bound for $f'(z)$ is attained for the function $f(z)$ defined by

$$-\frac{zf'(z)}{f(z)} = p_3(z)$$

while its lower bound is attained for the function $f(z)$ defined by

$$-\frac{zf'(z)}{f(z)} = p_4(z) \quad ,$$

$p_3(z)$, $p_4(z)$ being extremal for Theorems 5.2.1 and 5.2.2 respectively.

5.4 The class $\Sigma^*(\alpha)$ of meromorphic strongly starlike functions of order α .

As is well-known, a function $f(z) \in \mathbb{N}$ is univalent starlike in Δ if it satisfies the condition

$$|\arg\{\frac{zf'(z)}{f(z)}\}| < \frac{\pi}{2} \quad , \quad z \in \Delta \quad .$$

This condition is refined to define the class

$$S^*(\alpha) = \{f(z) \in \mathbb{N} \ ; \ |\arg\{\frac{zf'(z)}{f(z)}\}| < \alpha\frac{\pi}{2} \ , \ 0 < \alpha \le 1 \ , \ z \in \Delta\}$$

of strongly starkike functions of order α . This class was first studied by Brannan and Kirwan [11] and Stankiewicz [86] . In [11] , some distortion bounds for $S^*(\alpha)$ were given and the authors made use of properties of $S^*(\alpha)$ to study the coefficient behaviour of bounded univalent convex functions. Other geometric properties of $S^*(\alpha)$, including a geometric characterisation, were fully investigated by Stankiewicz [86] , [87] . In a following paper, Brannan, Clunie and Kirwan [12] gave sharp bounds for $|a_2|$ and $|a_3|$ for

$f(z) = z+a_2z^2+a_3z^3+\ldots\in S^*(\alpha)$. Goel and Singh [23] continued the study obtaining sharp bound for $|a_4|$ and partial results for $|a_5|$. The complete solution for this coefficient problem has not been found. In the same paper, [12] , Brannan, Clunie and Kirwan introduced a class of functions parallel to $S^*(\alpha)$, namely, the class $\sum^*(\alpha)$ of functions $g(z) = 1/z+b_0+b_1z+\ldots$ which are meromorphic and univalent in the unit disc Δ and satisfy there the condition

$$(5.4.1) \qquad (1-\frac{\alpha}{2})\pi < \arg\{\frac{zg'(z)}{g(z)}\} < (1+\frac{\alpha}{2})\pi \quad .$$

$\sum^*(1)$ is the well-known class $\sum^*$ of meromorphic univalent starlike functions. For $g(z) = 1/z+b_0+b_1z+\ldots\in\sum^*(\alpha)$, Brannan, Clunie and Kirwan [12] showed

$$|b_n| \le \frac{2\alpha}{n+1} \quad ,$$

with equality for a fixed n if and only if

$$\frac{zg'(z)}{g(z)} = -\left(\frac{1+\varepsilon z^{n+1}}{1-\varepsilon z^{n+1}}\right)^{\alpha} \quad , |\varepsilon| = 1 \quad .$$

In this section, we establish the distortion theorem, the covering theorem, the radius of convexity for $\sum^*(\alpha)$ and some relationship between $\sum^*(\alpha)$ and the class $\sum^*_\beta$ of meromorphic starlike functions of order β .

We first observe that condition (5.4.1) is equivalent to

$$(5.4.2) \qquad \frac{zg'(z)}{g(z)} = -p(z)^{\alpha} \quad , \quad p(z)\in P \; , \quad z\in\Delta \quad .$$

In view of this representation, the bounds for $|g'(z)|$ and the radius of convexity of $\sum^*(\alpha)$ can be obtained by solving the extremal problem

$$\min_{p(z)\in P}\ \min_{|z|=r<1} \mathrm{Re}\{p(z)^{\alpha}-\alpha\frac{zp'(z)}{p(z)}\} .$$

Again, using Dieudonné's lemma and writing $p(z) = Re^{i\theta}$, this problem becomes one which involves minimising a function, $S(R,\theta)$, of two variables in a domain D and on its boundary. However, an argument similar to that of Theorem 5.2.1 is not suitable in this case, as $S(R,\theta)$ now involves terms in $\cos\alpha\theta$ and $\cos\theta$. Instead, we employ the classical criterion for finding minima of functions of two variables to obtain the relative minima of $S(R,\theta)$ in D and then compare them with the values on the boundary ∂D . In fact, we shall see that $S(R,\theta)$ has only one minimum in D .

5.4.1 Theorem. *If* $p(z) \in P$, $0 < \alpha \leq 1$, *then on* $|z| = r < 1$,

$$\mathrm{Re}\{p(z)^{\alpha}-\alpha\frac{zp'(z)}{p(z)}\} \leq \frac{1-(1-2\alpha)r^2}{1-r^2} , \tag{5.4.3}$$

$$\mathrm{Re}\{p(z)^{\alpha}-\alpha\frac{zp'(z)}{p(z)}\} \geq \frac{1-(1+2\alpha)r^2}{1-r^2} . \tag{5.4.4}$$

The results are sharp.

Proof. From the representation $p(z) = [1+w(z)]/[1-w(z)]$, $w(z) \in B$, we may write

$$\mathrm{Re}\{p(z)^{\alpha}-\alpha\frac{zp'(z)}{p(z)}\} = \mathrm{Re}\{p(z)^{\alpha}- \frac{2\alpha zw'(z)}{1-w(z)^2}\}$$

$$\leq \mathrm{Re}\{p(z)^{\alpha}- \frac{2\alpha w(z)}{1-w(z)^2}\}+2\alpha\frac{r^2-|w(z)|^2}{(1-r^2)|1-w(z)^2|} , \text{ from } (1.2.4)$$

$$= \mathrm{Re}\{p(z)^{\alpha}- \frac{\alpha}{2}[p(z)- \frac{1}{p(z)}]\}+ \frac{\alpha}{2}\,\frac{r^2|p(z)+1|^2-|p(z)-1|^2}{(1-r^2)|p(z)|} . \tag{5.4.5}$$

Put $p(z) = Re^{i\theta}$ and denote the right-hand side of (5.4.5) by $S(R,\theta)$, then

$$S(R,\theta) = R^{\alpha}\cos\alpha\theta - \frac{\alpha}{2}\left(R - \frac{1}{R} - 2\frac{1+r^2}{1-r^2}\right)\cos\theta - \frac{\alpha}{2}\left(\frac{1}{R} + R\right) ,$$

from which we deduce that $\partial S/\partial R = 0$ and $\partial S/\partial\theta = 0$ are equivalent to the system of equations

$$(5.4.6) \qquad F(R,\theta) \equiv 2R^{\alpha-1}\cos\alpha\theta - \left(1+\frac{1}{R^2}\right)\cos\theta - 1 + \frac{1}{R^2} = 0 ,$$

$$(5.4.7) \qquad G(R,\theta) \equiv 2R^{\alpha}\sin\alpha\theta + \left(\frac{1}{R} - R + 2\frac{1+r^2}{1-r^2}\right)\sin\theta = 0 .$$

Since $p(z)$ maps $|z| \le r$ into the disc $|p(z)-a| \le d$, where

$$(5.4.8) \qquad a = \frac{1+r^2}{1-r^2} , \quad d = \frac{2r}{1-r^2} ,$$

we maximise $S(R,\theta)$ over the domain

$$D = \{(R,\theta) ; a-d < R < a+d , -\psi(R) < \theta < \psi(R)\}$$

and on its boundary ∂D, where

$$\psi(R) = \arccos\frac{R^2+a^2-d^2}{2aR} , \quad 0 \le \psi(R) \le \psi(1) .$$

We want to show now that the only solution of the system of equations (5.4.6) and (5.4.7) is $(R,\theta) = (1,0)$. We observe first of all that $S(R,\theta) = S(R,-\theta)$. Hence it suffices to consider only $\theta \in [0,\pi/2)$. It is clear that $(1,0)$ is a solution of the system. For $R = 1$, $\theta = \theta_1 \ne 0$, $G(1,\theta_1) \ge 4\sin\alpha\theta_1/(1-r^2) > 0$. Hence $(1,\theta_1)$ is not a solution. For $R = R_1 \ne 1$, $\theta = 0$, $F(R_1,0) = 2(R_1^{\alpha-1}-1) \ne 0$. Hence

$(R_1,0)$ is not a solution. For $R = R_2 < 1$, $\theta = \theta_2 \neq 0$, it follows directly from (5.4.7) that $G(R_2,\theta_2) > 0$. Hence (R_2,θ_2) is not a solution. The last case we have to consider is $R = R_3 > 1$, $\theta = \theta_3 \neq 0$. For this case,

$$\frac{1}{R} - R + 2\frac{1+r^2}{1-r^2} > \frac{1}{a+d} - (a+d) + 2\frac{1+r^2}{1-r^2} = 2\frac{1-r}{1+r} > 0 \quad .$$

Hence $G(R_3,\theta_3) > 0$, which shows that (R_3,θ_3) is not a solution. At the point $(R,\theta) = (1,0)$ and for $\alpha \in (0,1)$, we have

$$\frac{\partial^2 S}{\partial R^2} = \alpha(\alpha-1) < 0 \quad ,$$

$$\frac{\partial^2 S}{\partial R^2} \cdot \frac{\partial^2 S}{\partial \theta^2} - \left(\frac{\partial^2 S}{\partial R \partial \theta}\right)^2 = \alpha^3(1-\alpha) + \alpha^2(1-\alpha)\frac{1+r^2}{1-r^2} > 0 \quad .$$

Thus

$$\max_{(R,\theta)\in D} S(R,\theta) = S(1,0) = \frac{1-(1-2\alpha)r^2}{1-r^2} \quad .$$

We now look at the case $(R,\theta) \in \partial D$. By writing $p(z) = a+u+iv$, we have, at a point on the boundary ∂R, $d^2 = u^2+v^2$. Thus

$$r^2|p(z)+1|^2 - |p(z)-1|^2 = (1-r^2)(d^2-u^2-v^2) = 0 \quad .$$

This together with (5.4.5) yield, for $(R,\theta) \in \partial D$,

$$S(R,\theta) = R^\alpha \cos\alpha\theta - \frac{\alpha}{2}\left(R - \frac{1}{R}\right)\cos\theta$$

$$(5.4.9) \qquad = R^\alpha \cos\alpha \arccos\left(\frac{R^2+1}{2aR}\right) - \frac{\alpha}{2}\,\frac{R^2-R^{-2}}{2a} \;, \text{ as } \cos\theta = \frac{R^2+a^2-d^2}{2aR}$$

$$= T(R) \quad .$$

We next show that $T(R) \le S(R,\theta)$ for $(R,\theta) \in D$, $a-d \le R \le a+d$.

Indeed, since

$$R - \frac{1}{R} - 2\frac{1+r^2}{1-r^2} \le a+d - \frac{1}{a+d} - 2\frac{1+r^2}{1-r^2} = -2\frac{1-r}{1+r} < 0 \quad ,$$

we have

$$S(R,\theta) \ge R^{\alpha}\cos\alpha\theta - \frac{\alpha}{2}[(R - \frac{1}{R} - 2\frac{1+r^2}{1-r^2})\frac{R^2+1}{2aR} + \frac{1}{R} + R]$$

$$(5.4.10) \qquad = R^{\alpha}\cos\alpha\theta - \frac{\alpha}{2} \cdot \frac{R^2 - R^{-2}}{2a} \quad .$$

Comparing (5.4.9) and (5.4.10) , the assertion follows. Consequently,

$$\max_{(R,\theta)\in D\cup\partial D} S(R,\theta) = \frac{1-(1-2\alpha)r^2}{1-r^2}$$

and this proves (5.4.3) . Using the same argument, we can show that

$$\mathrm{Re}\{p(z)^{\alpha} - \alpha\frac{zp'(z)}{p(z)}\} \ge R^{\alpha}\cos\alpha\theta - \frac{\alpha}{2}(R - \frac{1}{R} + 2\frac{1+r^2}{1-r^2})\cos\theta + \frac{\alpha}{2}(\frac{1}{R} + R) = H(R,\theta)$$

and

$$\min_{(R,\theta)\in D\cup\partial D} H(R,\theta) = H(1,0) = \frac{1-(1+2\alpha)r}{1-r^2} \quad ,$$

which proves (5.4.4).

To see that the results are sharp,we consider the function $p_0(z) = [1+w_0(z)]/[1-w_0(z)]$, where $w_0(z) = z(z-c)/(1-cz)$ with c determined by the condition $\mathrm{Re}\{[1+w_0(z)]/[1-w_0(z)]\} = 1$ at $z = r$ for (5.4.3) and at $z = -r$ for (5.4.4) .

5.4.2 <u>Corollary</u>. *Let* $f(z) \in \sum^{*}(\alpha)$; *then on* $|z| = r < 1$,

$$(5.4.11) \qquad \{r \exp\int_0^r [(\frac{1+t}{1-t})^{\alpha} - 1]\frac{dt}{t}\}^{-1} \le |f(z)| \le \{r \exp\int_0^r [(\frac{1-t}{1+t})^{\alpha} - 1]\frac{dt}{t}\}^{-1} \quad ,$$

$$(5.4.12) \qquad \frac{(1-r^2)^{\alpha}}{r^2} \le |f'(z)| \le \frac{1}{r^2(1-r^2)^{\alpha}} \quad .$$

Proof. We first note that a function g(z) belongs to $S^*(\alpha)$ if and only if

$$\frac{zg'(z)}{g(z)} = p(z)^\alpha \quad , \quad p(z) \in P \quad .$$

Hence the structural formula for $S^*(\alpha)$ may be derived to be

$$(5.4.13) \qquad g(z) = z \exp \int_0^z [p(\xi)^\alpha - 1]\frac{d\xi}{\xi} \quad .$$

It follows from (5.4.8) that, on $|z| = r < 1$,

$$(5.4.14) \qquad \left(\frac{1-r}{1+r}\right)^\alpha \le \mathrm{Re}\{p(z)^\alpha\} \le \left(\frac{1+r}{1-r}\right)^\alpha \quad .$$

Since $f(z) \in \Sigma^*(\alpha)$ if and only if $1/f(z) \in S^*(\alpha)$, (5.4.11) may be obtained immediately from (5.4.13) and (5.4.14) . To prove (5.4.12) , we make use of (5.3.2) and (5.4.3) to get

$$r\frac{\partial}{\partial r}\log|z^2 f'(z)| \ge -\frac{2\alpha r^2}{1-r^2} \quad ,$$

which yields on integrating both sides

$$\log|z^2 f'(z)| \ge \log(1-r^2)^\alpha \quad ,$$

that is, $|f'(z)| \ge r^{-2}(1-r^2)^\alpha$. The upper bound for $|f'(z)|$ is proved similarly.

The inequalities in (5.4.11) are sharp for the function f(z) defined by

$$\frac{zf'(z)}{f(z)} = -\left(\frac{1+z}{1-z}\right)^\alpha \quad ,$$

while those in (5.4.12) are sharp for the function f(z) defined by

$$\frac{zf'(z)}{f(z)} = -\, p_0(z)^{\alpha} \quad ,$$

where $p_0(z)$ is extremal for Theorem 5.4.1.

5.4.3 Corollary. *The complement of the image of the unit disc under a function* $f(z) \in \sum^*(\alpha)$ *contains the closed disc*

$$|w| \le \exp\left[\sum_{k=0}^{\infty} \frac{2\alpha}{(2k+1)(2k+1-\alpha)}\right] .$$

Proof. Let $r \to 1$ in (5.4.11) we find

$$|f(z)| \ge \left\{\exp \int_0^1 \left[\left(\frac{1+t}{1-t}\right)^{\alpha} - 1\right]\frac{dt}{t}\right\}^{-1} .$$

It was shown previously by Brannan and Kirwan [11] that

$$\int_0^1 \left[\left(\frac{1+t}{1-t}\right)^{\alpha} - 1\right]\frac{dt}{t} = \sum_{k=0}^{\infty} \frac{2\alpha}{(2k+1)(2k+1-\alpha)} .$$

Hence the assertion follows.

5.4.4 Corollary. *The radius of convexity of* $\sum^*(\alpha)$ *is* $(1+2\alpha)^{-\frac{1}{2}}$.

Proof. For $f(z) \in \sum^*(\alpha)$, we may write

$$-\left[1+ \frac{zf''(z)}{f'(z)}\right] = p(z)^{\alpha} - \alpha\frac{zp'(z)}{p(z)} \quad , \quad p(z) \in P .$$

Hence an application of (5.4.4) gives , on $|z| = r$,

$$-\mathrm{Re}\left\{1+ \frac{zf''(z)}{f'(z)}\right\} \ge \frac{1-(1+2\alpha)r^2}{1-r^2}$$

which is positive for $r < (1+2\alpha)^{-\frac{1}{2}}$.

We next study the inclusion relationship between the class Σ_α^* of meromorphic starlike functions of order α and the class $\Sigma^*(\beta)$ of meromorphic strongly starlike functions of order β. In the limiting cases $\alpha = 0$, $\beta = 1$, both classes reduce to Σ^*.

We recall that for $f(z) \in \Sigma_\alpha^*$, the following inequality follows from (1.2.1) and (1.2.2) with $A = 1-2\alpha$, $B = -1$.

$$(5.4.15) \qquad |\arg\{\frac{zf'(z)}{f(z)}\}| \le \pi+\sin^{-1}\frac{2(1-\alpha)r}{1+(1-2\alpha)r^2}, \qquad |z| = r.$$

5.4.5 Theorem.

(i) $\Sigma_\alpha^* \subseteq \Sigma^*(\beta)$ *in* $|z| < \sin(\beta\pi/2)\{1-\alpha+[(1-\alpha)^2-(1-2\alpha)\sin^2(\beta\pi/2)]^{\frac{1}{2}}\}^{-1}$,

(ii) $\Sigma^*(\beta) \subseteq \Sigma_\alpha^*$ *in* $|z| < (1-\alpha^{1/\beta})/(1+\alpha^{1/\beta})$.

The results are sharp.

Proof. (i) For $f(z) \in \Sigma_\alpha^*$, we have from (5.4.15) that

$$|\arg\{\frac{zf'(z)}{f(z)}\}| \le \pi+\sin^{-1}\frac{2(1-\alpha)r}{1+(1-2\alpha)r^2}, \qquad |z| = r.$$

Hence $f(z)$ will be in $\Sigma^*(\beta)$ if

$$\pi+\sin^{-1}\frac{2(1-\alpha)r}{1+(1-2\alpha)r^2} < \pi+\beta\frac{\pi}{2},$$

or, equivalently,

$$F(r) \equiv \sin(\beta\pi/2)(1-2\alpha)r^2-2(1-\alpha)r+\sin(\beta\pi/2) > 0.$$

Now $F(0) = \sin(\beta\pi/2) > 0$, $F(1) = -2(1-\alpha)(1-\sin(\beta\pi/2)) < 0$. Hence $F(r)$ has a zero in $(0,1)$. In fact, the only zero in $(0,1)$ of $F(r)$ is $r_0 = \sin(\beta\pi/2)\{1-\alpha+[(1-\alpha)^2-(1-2\alpha)\sin^2(\beta\pi/2)]^{\frac{1}{2}}\}^{-1}$.

Equality occurs in (5.4.15) for the function $f(z)$ defined by

$$\frac{zf'(z)}{f(z)} = -\frac{1+(1-2\alpha)\varepsilon z}{1-\varepsilon z} \quad , \quad |\varepsilon| = 1 \quad ;$$

hence this function is extremal for part (i) .

(ii) In view of (5.4.14) we have, for $f(z) \in \sum^*(\beta)$,

$$-\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} = \mathrm{Re}\{p(z)^\beta\} \geq (\frac{1-r}{1+r})^\beta \quad , \quad |z| = r \quad .$$

Hence, $f(z) \in \sum_\alpha^*$ if

$$(\frac{1-r}{1+r})^\beta > \alpha \quad .$$

This is equivalent to $r < (1-\alpha^{1/\beta})/(1+\alpha^{1/\beta})$.

The function $f(z)$ for which

$$\frac{zf'(z)}{f(z)} = -(\frac{1+z}{1-z})^\beta$$

serves as an extremal function.

CHAPTER 6

THE STARLIKENESS OF CERTAIN CLOSE-TO-STARLIKE FUNCTIONS

6.1 Introduction

As is well-known [45], the radius of starlikeness of close-to-starlike functions, that is, functions $f(z)$ for which $\mathrm{Re}\{e^{i\theta} f(z)/g(z)\} > 0$ in Δ, where $0 \le \theta < 2\pi$, $g(z) \in S^*$, is $2-\sqrt{3}$. When $g(z)$ is restricted to the class S^c, MacGregor [45] showed that $f(z)$ is starlike in $|z| < 1/3$. For $g(z)$ arbitrary in the whole class S, Krzyż and Reade [36] proved that $f(z)$ is starlike in $|z| < 2-\sqrt{3}$. Since $S^c \subseteq S^*_{\frac{1}{2}}$ and $S^c \subseteq R_{\frac{1}{2}}$, Ratti [69] considered the problem for the cases $g(z) \in S^*_\alpha$, $g(z) \in R$ and $g(z) \in R_{\frac{1}{2}}$. In particular, he found that the radii of starlikeness of $f(z)$ for the cases $g(z) \in S^*_{\frac{1}{2}}$ and $g(z) \in R_{\frac{1}{2}}$ are the same as that for $g(z) \in S^c$. The above authors also determined the radius of starlikeness of $f(z)$ when the condition $\mathrm{Re}\{f(z)/g(z)\} > 0$ is replaced by $|f(z)/g(z) - 1| < 1$ in Δ (see [46], [36], [69]).

In an attempt to generalise these results, Shah [83] considered the problem of determining the radius of starlikeness of functions $f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \dots$, $k = 1,2,3,\dots$, which satisfy the condition

$$\mathrm{Re}\left\{\frac{f(z)}{\lambda f(z) + (1-\lambda)g(z)}\right\} > 0 \quad , z \in \Delta ,$$

or

$$\left|\frac{f(z)}{\lambda f(z) + (1-\lambda)g(z)} - 1\right| < 1 , \ z \in \Delta ,$$

for $0 \le \lambda < 1$, $g(z)$ varying in each of the subclasses considered by

Ratti which we have mentioned above. However, the results given by Shah are sharp only when $\lambda = 0$, that is, in the known cases.

It is the purpose of this chapter to establish the radius of starlikeness of functions $f(z) \in \mathcal{N}$ and satisfying the condition

(6.1.1) $$\left|\frac{f(z)}{\lambda f(z) + (1-\lambda)g(z)} - \gamma\right| < \gamma , \quad z \in \Delta ,$$

where $\gamma \geq 1$, $0 \leq \lambda < 1$ and $g(z)$ belongs to each of the following cases:

(i) $g(z) \in \mathcal{R}$,

(ii) $g(z) \in \mathcal{R}_{\frac{1}{2}}$,

(iii) $g(z) \in S_{\alpha}^{*}$, with $g(z) \in S^{c}$ as a special case, and

(iv) $g(z) \in S$.

Letting $\gamma \to \infty$ we obtain the radius of starlikeness of $f(z)$ which satisfy $\mathrm{Re}\{f(z)/[\lambda f(z) + (1-\lambda)g(z)]\} > 0$ in Δ . All the bounds established are best possible and generalise the results by MacGregor [45] , [46] , Ratti [69] , Krzyż and Reade [36] when we put $\lambda = 0$ and let $\gamma \to \infty$ or $\gamma = 1$. Furthermore, we shall briefly look at the problem of determining the radius of convexity of functions $f(z) \in \mathcal{N}$ for which

$$\left|\frac{f'(z)}{\lambda f'(z) + (1-\lambda)g'(z)} - \gamma\right| < \gamma , \quad \gamma \geq 1 , \ z \in \Delta ,$$

where $g(z)$ belongs to some subclass of $\mathcal{N}$ or S .

Our approach to the problems under consideration is based upon solving the extremal problem

(6.1.2) $$\max_{p(z)\in P} \ \max_{|z|=r<1} \mathrm{Re}\left\{\frac{zp'(z)}{p(z) + \mu}\right\} , \quad \mu \in (-1,1] .$$

Since the function $zp'(z)/[p(z)+\mu]$ is not regular in the entire unit disc for $\mu < 0$, we have to restrict z to some smaller disc, for example, $|z| < (1-\mu)/(1+\mu)$. However, this restriction can be removed at the expense of some range of λ .

In the terminology of this thesis, (6.1.2) is merely a particular case of the following problem, which we shall deal with in the next section,

$$\text{(6.1.3)} \qquad \max_{|z|=r<1} \operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} ,$$

where $p(z)$ now varies in $P(A, B)$ with $A \in (-1,\infty)$, $-1 \le B < 1$ and $B < A$. Hence the work of this chapter may also be considered as a continuation of the study of extremal problems over $P(A, B)$ which we have partly carried out in Chapters 1 and 5 .

In the final section of this chapter, we examine the convexity of certain close-to-starlike functions. In particular, the radius of convexity of a special subclass is given.

Theorems 6.3.1-3, 6.4.1-6 of this chapter were published in Anh and Tuan [4] .

6.2 The functional $\operatorname{Re}\{zp'(z)/p(z)\}$ over $P(A, B)$ with $-1 < A < \infty$

The fact that μ varies in the range $(-1,1]$ for the extremal problem (6.1.2) gives rise to the consideration of A in $(-1,\infty)$ for the problem (6.1.3) . We first consider A in the interval $(-1,1]$. In this case, Theorem 5.2.1 yields immediately

6.2.1 Corollary. *If* $p(z) \in P(A, B)$, $-1 \le B < A \le 1$,*then on* $|z| = r < 1$,

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \le \begin{cases} G_1(A,B;r) & , \quad R_5' \ge R_6 \ , \\ G_2(A,B;r) & , \quad R_6 \ge R_5' \ , \end{cases}$$

where $G_1(A,B;r) = \dfrac{(A-B)r}{(1+Ar)(1+Br)}$,

$$G_2(A,B;r) = \frac{A+B}{A-B} - \frac{2}{(A-B)(1-r^2)}\{[(1+A)(1+B)(1-Ar^2)(1-Br^2)]^{\frac{1}{2}} - (1-ABr^2)\} \ ,$$

$$R_5' = [(1+A)(1-Ar^2)/(1+B)(1-Br^2)]^{\frac{1}{2}} \quad , R_6 = (1+Ar)/(1+Br) \ .$$

For A in the interval $[1,\infty)$, we have

6.2.2 <u>Theorem</u>. *Let* $p(z) \in P(A, B)$, $1 \le A < \infty$, $-1 \le B < 0$ *and* r_0 *be the smallest root in* $(0,1)$ *of the equation*

(6.2.1) $ABr^4 - 2ABr^3 - [1+2(A+B)+AB]r^2 - 2r + 1 = 0$.

Then, on $|z| = r < 1$,

(i) *for* $r_0 < 1/A$,

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \le \begin{cases} G_1(A,B;r) & , \quad 0 < r \le r_0 \ , \\ G_2(A,B;r) & , \quad r_0 \le r < 1/A \ . \end{cases}$$

(ii) *for* $r_0 \ge 1/A$,

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \le G_1(A,B;r) \ , \ 0 < r < 1/A \ ,$$

$G_1(A,B;r)$, $G_2(A,B;r)$ *being as given in Corollary* 6.2.1 . *The result is sharp.*

<u>Proof</u>. For $p(z) \in P(A, B)$, we have

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} = \operatorname{Re}\left\{\frac{(A-B)zw'(z)}{[1+Aw(z)][1+Bw(z)]}\right\} \quad , \quad w(z) \in B$$

$$(6.2.2) \qquad \le (A-B)\{\mathrm{Re}\{\frac{w(z)}{[1+Aw(z)][1+Bw(z)]}\} + \frac{r^2 - |w(z)|^2}{(1-r^2)|1+Aw(z)||1+Bw(z)|}\} ,$$

from (1.2.4) .

Since $A \ge 1$, the function $1+Aw(z)$ may vanish in Δ . To avoid this, we require $|z| < 1/A$. Now, define $w_1(z) = [1+Bw(z)]/[1+Aw(z)]$. Then $w_1(z)$ maps $|z| \le r$ into the disc $|w_1(z)-\alpha| \le \delta$, where

$$\alpha = \frac{1-ABr^2}{1-A^2r^2} \quad , \quad \delta = \frac{(A-B)r}{1-A^2r^2} \ .$$

In terms of $w_1(z)$, (6.2.2) becomes

$$(6.2.3) \ \mathrm{Re}\{\frac{zp'(z)}{p(z)}\} \le \frac{A+B}{A-B} - \frac{1}{A-B}\mathrm{Re}\{Aw_1(z)+\frac{B}{w_1(z)}\} + \frac{r^2|B-Aw_1(z)|^2-|w_1(z)-1|^2}{(A-B)(1-r^2)|w_1(z)|} \ .$$

Put $w_1(z) = \alpha+u+iv$, $R = |w_1(z)|$ and denote the right-hand side of (6.2.3) by $S(u,v)$, then

$$S(u,v) = \frac{A+B}{A-B} + \frac{1}{A-B}[- A(\alpha+u)- \frac{B(\alpha+u)}{R^2} + \frac{1-A^2r^2}{1-r^2} . \frac{\delta^2-u^2-v^2}{R}] \ .$$

This yields

$$\frac{\partial S}{\partial v} = - \frac{1}{A-B} \cdot \frac{v}{R^4} T(u,v) \ ,$$

where

$$T(u,v) = -2B(\alpha+u) + \frac{1-A^2r^2}{1-r^2} [2R^3 + R(\delta^2-u^2-v^2)]$$

which is always positive as $B < 0$, $r < 1/A$ and

$$\alpha+u \ge \alpha-\delta = \frac{1+Br}{1+Ar} > 0 \ .$$

Hence the maximum of $S(u,v)$ on the disc $|w_1(z)-\alpha| \leq \delta$ is attained when $v = 0$, $u \in [-\delta,\delta]$. Now,

$$S(u,0) = \frac{A+B}{A-B} + \frac{1}{A-B}\left[-\frac{(1+A)(1-Ar^2)}{1-r^2}(\alpha+u) - \frac{(1+B)(1-Br^2)}{1-r^2}\cdot\frac{1}{\alpha+u} + 2\frac{1-ABr^2}{1-r^2}\right].$$

Hence

$$\frac{dS(u,0)}{du} = \frac{1}{A-B}\left[-\frac{(1+A)(1-Ar^2)}{1-r^2} + \frac{(1+B)(1-Br^2)}{1-r^2}\cdot\frac{1}{(\alpha+u)^2}\right]$$

which vanishes at

$$u_0 = \left[\frac{(1+B)(1-Br^2)}{(1+A)(1-Ar^2)}\right]^{\frac{1}{2}} - \alpha .$$

Thus if $u_0 \in [-\delta,\delta]$, the maximum of $S(u,0)$ occurs at $u = u_0$, its value being $S(u_0,0) = G_2(A,B;r)$. It is easy to check that

$$(\alpha+u_0)^2 < \frac{1-Br^2}{1-Ar^2} < \frac{1-Br}{1-Ar} < \left(\frac{1-Br}{1-Ar}\right)^2 = (\alpha+\delta)^2 .$$

Thus $u_0 < \delta$. However, u_0 is not always greater than $-\delta$. For $u_0 \leq -\delta$, which is equivalent to $R_5' \geq R_6$ in Corollary 6.2.1, the maximum of $S(u,0)$ occurs at $u = -\delta$. A simple calculation shows $S(-\delta,0) = G_1(A,B;r)$. The transition point between the two cases is determined by the equation

$$\frac{(1+B)(1-Br^2)}{(1+A)(1-Ar^2)} = \left(\frac{1+Br}{1+Ar}\right)^2 ,$$

or equivalently,

$$F(r) \equiv ABr^4 - 2ABr^3 - [1+2(A+B)+AB]r^2 - 2r + 1 = 0 .$$

Now, $F(0) = 1$, $F(1) = -2(1+A)(1+B) < 0$. Hence $F(r)$ has a zero in $(0,1)$.

Denoting by r_0 its smallest zero in this interval and taking into account the condition $r < 1/A$, the result follows.

6.2.3 Corollary. *If* $p(z) \in \mathcal{P}$, $-1 < \mu \le 1$, *then on* $|z| = r < \min\{1,(1+\mu)/(1-\mu)\}$,

$$(6.2.4) \qquad \mathrm{Re}\{\frac{zp'(z)}{p(z)+\mu}\} \le \frac{2r}{[1+\mu+(1-\mu)r](1-r)} .$$

Proof. For $p(z) \in \mathcal{P}$, we may write

$$\frac{zp'(z)}{p(z)+\mu} = \frac{2}{1+\mu} \cdot \frac{zw'(z)}{[1+Aw(z)][1-w(z)]} , \quad w(z) \in \mathcal{B} ,$$

where $A = (1-\mu)/(1+\mu)$. For $\mu \in [0,1]$, that is , $A \in [0,1]$, the condition $R_5' \ge R_6$ of Corollary 6.2.1 with $B = -1$ becomes $(1-r^2)(1-Ar^2) \ge 0$, which always holds for $r < 1$. Hence, the second case $R_6 \ge R_5'$ does not exist and (6.2.4) with $\mu \in [0,1]$ follows from Corollary 6.2.1. For $\mu \in (-1,0]$, that is, $A \in [1,\infty)$, the only root in (0,1) of the equation (6.2.1) is $r_0 = 1/\sqrt{A} \ge 1/A$. Thus (6.2.4) with $\mu \in (-1,0]$ follows from part (ii) of Theorem 6.2.2.

Equality in (6.2.4) occurs for the function $p(z) = (1+z)/(1-z)$ at $z = r$.

6.3 The starlikeness of certain close-to-starlike functions

Corollary 6.2.3 will now be used to determine the radius of starlikeness of functions $f(z)$ defined by condition (6.1.1). To simplify the statements of the following theorems, we shall denote by $\mathcal{P}(\gamma)$ the class of functions $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ for which

$$|p(z) - \gamma| < \gamma \quad , \gamma \ge 1 \quad , \quad z \in \Delta \quad .$$

6.3.1 <u>Theorem.</u> *Let* $f(z) \in \mathcal{N}$ *be such that* $f(z)/[\lambda f(z)+(1-\lambda)g(z)] \in P(\gamma)$, *where* $g(z) \in R$, $0 \le \lambda < (1+\sqrt{3}+1/2\gamma)/(2+\sqrt{3})$. *Then the radius of starlikeness* σ_1 *of* $f(z)$ *is given by the only root in* $(0,1)$ *of the equation*

$$Ar^3 + (2+3A)r^2 + 3r - 1 = 0 ,$$

where $A = [(1+\lambda)\gamma-1]/(1-\lambda)\gamma$.

<u>Proof</u>. Define

$$\psi(z) = 1 - \frac{f(z)}{\gamma[\lambda f(z)+(1-\lambda)g(z)]} . \tag{6.3.1}$$

Then $|\psi(z)| < 1$ in Δ and $\psi(0) = 1-1/\gamma = \beta$. Let $w(z)=[\psi(z)-\beta]/[1-\beta\psi(z)]$. It is clear that $w(z) \in B$ and

$$\psi(z) = \frac{w(z) + \beta}{1 + \beta w(z)} . \tag{6.3.2}$$

From (6.3.1) and (6.3.2) we deduce that

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} - \frac{1+\beta}{1-\lambda} \cdot \frac{zw'(z)}{[1+Aw(z)][1-w(z)]} , \tag{6.3.3}$$

where $A = (\beta+\lambda)/(1-\lambda) = [(1+\lambda)\gamma-1]/(1-\lambda)\gamma$, provided $1-\lambda[1-w(z)]/[1+\beta w(z)] \neq 0$. Since $|w(z)| \le r$ on $|z| = r$ by Schwarz's lemma, it follows that $1-\lambda[1-w(z)]/[1+\beta w(z)] \neq 0$ if, in particular, $|z| < 1/A$. Equation (6.3.3) can also be rewritten in terms of $p(z) \in P$ as

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu} , \tag{6.3.4}$$

where $\mu = (1-A)/(1+A)$, that is , $A = (1-\mu)/(1+\mu)$. Now, as $g(z) \in R$, we may write $g(z) = zq(z)$, for some $q(z) \in P$. This implies

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zq'(z)}{q(z)} .$$

In view of (1.2.13) we get

$$\mathrm{Re}\{\frac{zg'(z)}{g(z)}\} \geq \frac{1-2r-r^2}{1-r^2} \quad , \quad |z| = r < 1 \quad .$$

This result together with (6.2.4) and (6.3.4) yield , on $|z| = r$,

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} \geq \frac{1-3r-(2+3A)r^2-Ar^3}{(1-r)(1+Ar)} \quad .$$

For the cubic polynomial

$$F(r) = Ar^3 + (2+3A)r^2 + 3r - 1 \ ,$$

we have $F(0) = -1 < 0$, $F(1) = 4+4A > 0$, $F(1/A) = (3+6A-A^2)/A^2$. Thus the equation $F(r) = 0$ has exactly one root in $(0,1)$ which is in the range $(0,1/A)$ if $A < 3+2\sqrt{3}$, that is, if $\lambda<(1+\sqrt{3}+1/2\gamma)/(2+\sqrt{3})$.

The result is sharp for the function

$$f(z) = \frac{1-z}{1+Az} \cdot \frac{z(1-z)}{1+z} \quad .$$

When $\lambda = 0$, $f(z)$ is starlike in $|z| < \sqrt{5}-2$ if $\gamma \to \infty$ and in $|z| < (\sqrt{17}-3)/4$ if $\gamma = 1$ as previously shown by Ratti [69, Theorems 1, 4].

6.3.2 <u>Theorem</u>. *Let* $f(z) \in \mathrm{N}$ *be such that* $f(z)/[\lambda f(z)+(1-\lambda)g(z)] \in P(\gamma)$, *where* $g(z) \in R_{\frac{1}{2}}$. *Let* r_1 *be the smallest root in* $(0,1)$ *of the equation*

$$(1+2A+9A^2)r^4 + 2(1+12A+3A^2)r^3 + (13+10A+A^2)r^2 + 4(1-A)r - 4 = 0 \ ,$$

$A = [(1+\lambda)\gamma-1]/(1-\lambda)\gamma$ *and* $r_2 = [\sqrt{2}(1+A)^{\frac{1}{2}}-1]/(1+2A)$. *Then the radius of starlikeness of* $f(z)$ *is*

$$\sigma_2 = \begin{cases} r_1 \ , & 0 \leq \lambda \leq 1/2\gamma \ , \\ r_2 \ , & 1/2\gamma \leq \lambda < (\sqrt{5}+1+1/\gamma)/(\sqrt{5}+3) \ . \end{cases}$$

Proof. Since $g(z) \in R_{\frac{1}{2}}$, there exists $q(z) \in P$ such that $g(z)/z = \frac{1}{2} + q(z)/2$. Consequently,

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zq'(z)}{q(z)+1} \quad , \quad z \in \Delta \quad .$$

Applying (1.2.11) with $\gamma = \frac{1}{2}$ to the right-hand side gives, on $|z| = r$,

$$\mathrm{Re}\{\frac{zg'(z)}{g(z)}\} \geq \begin{cases} 1/(1+r) \quad , \quad 0 < r \leq 1/3 \quad , \\ \\ 2[\sqrt{2}(1-r^2)^{\frac{1}{2}}-1]/(1-r^2) \quad , \quad 1/3 \leq r < 1 \quad . \end{cases}$$

This result together with (6.2.4) and (6.3.4) yield , for $|z| = r \leq 1/3$,

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} \geq \frac{1-2r-(1+2A)r^2}{(1-r)(1+Ar)} \tag{6.3.5}$$

and for $1/3 \leq r < 1$,

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} \geq -\frac{(1+A)r}{(1-r)(1+Ar)} + \frac{2[\sqrt{2}(1-r^2)^{\frac{1}{2}}-1]}{1-r^2} \quad ,$$

which implies the equation giving the radius of starlikeness of $f(z)$ to be

$$(1+2A+9A^2)r^4+2(1+12A+3A^2)r^3+(13+10A+A^2)r^2+4(1-A)r-4 = 0 \quad .$$

The only root in (0,1) of the numerator of the right-hand side of (6.3.5) is r_2 which is less than or equal to 1/3 if $A \geq 1$, that is, if $\lambda \geq 1/2\gamma$, and is in the range $(0,1/A)$ if $A < \sqrt{5}+2$, that is, if $\lambda < (\sqrt{5}+1+1/\gamma)/(\sqrt{5}+3)$. Thus $f(z)$ is starlike in $|z| < r_2$ if $1/2\gamma \leq \lambda < (\sqrt{5}+1+1/\gamma)/(\sqrt{5}+3)$. Now, for $0 \leq \lambda < 1/2\gamma$, we have $A \leq 1$; hence r_1 is in the interval $(0,1/A)$. The proof of the theorem is thus completed.

The result is sharp for

$$f(z) = \frac{1-z}{1+Az} \cdot \frac{z}{2}\left[1+\tfrac{1}{2}\left(\frac{1+ze^{i\theta}}{1-ze^{i\theta}} + \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right)\right] , \text{ for } 0 \le \lambda \le 1/2\gamma ,$$

$$f(z) = \frac{1-z}{1+Az} \cdot \frac{z}{1+z} , \text{ for } 1/2\gamma \le \lambda < (\sqrt{5}+1+1/\gamma)/(\sqrt{5}+3) ,$$

where θ satisfies the equation

$$\frac{1+r\cos\theta}{1+2r\cos\theta+r^2} = 2^{-\frac{1}{2}}(1-r^2)^{-\frac{1}{2}} .$$

When $\lambda = 0$, the cases $\gamma \to \infty$ and $\gamma = 1$ give Theorems 2 and 5 of Ratti [69] .

If now $g(z)$ belongs to S_α^* , $0 \le \alpha < 1$, then as given by (1.3.7),

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \ge \frac{1-(1-2\alpha)r}{1+r} , \quad |z| = r < 1 .$$

Consequently, an application of this result in conjunction with (6.2.4) and (6.3.4) will give

6.3.3 <u>Theorem</u>. *Let* $f(z) \in N$ *be such that* $f(z)/[\lambda f(z)+(1-\lambda)g(z)] \in P(\gamma)$, *where* $g(z) \in S_\alpha^*$. *Then the radius of starlikeness* σ_3 *of* $f(z)$ *is given by the smallest root in* $(0,1)$ *of the equation*

$$A(2\alpha-1)r^3 + (3A+2\alpha-2\alpha A)r^2 + (3-2\alpha)r - 1 = 0$$

for $0 \le \lambda < \lambda_0$, λ_0 *being determined from the condition* $\sigma_3 < 1/A$, $A = [(1+\lambda)\gamma-1]/(1-\lambda)\gamma$.

The result is sharp for the function

$$f(z) = \frac{1-z}{1+Az} \cdot \frac{z}{(1+z)^{2-2\alpha}} .$$

When $\lambda = 0$, the cases $\gamma \to \infty$ and $\gamma = 1$ correspond to Theorems 3 and 6 of Ratti [69].

6.3.4 <u>Remark</u>. Putting $\alpha = \frac{1}{2}$ in Theorem 6.3.3, we obtain immediately that the radius of starlikeness of $f(z) \in N$ for which $f(z)/[\lambda f(z)+(1-\lambda)g(z)] \in P(\gamma)$, $g(z) \in S^*_{\frac{1}{2}}$, is $[\sqrt{2}(1+A)^{\frac{1}{2}}-1]/(1+2A)$ for $0 \le \lambda < (\sqrt{5}+1+1/\gamma)/(\sqrt{5}+3)$. The bound is attained for the function

$$f(z) = \frac{1-z}{1+Az} \cdot \frac{z}{1+z} .$$

Since $S^c \subseteq S^*_{\frac{1}{2}}$ and the function $z/(1+z)$ extremal for the case $g(z) \in S^*_{\frac{1}{2}}$ also belongs to S^c, the above result is sharp for the case $g(z) \in S^c$.

When $\lambda = 0$, letting $\gamma \to \infty$ and $\gamma = 1$ we obtain Theorem 4 of MacGregor [45] and Theorem 4 of MacGregor [46] respectively.

6.3.5 <u>Theorem</u>. *Let* $f(z) \in N$ *be such that* $f(z)/[\lambda f(z)+(1-\lambda)g(z)] \in P(\gamma)$, *where* $g(z) \in S$. *Then the radius of starlikeness* σ_4 *of* $f(z)$ *is given by the only root in* $(0,1/3)$ *of the equation*

$$Ar^3 - 3Ar^2 - 3r + 1 = 0$$

for $0 \le \lambda < \lambda_1$, λ_1 *being determined from the condition* $\sigma_4 < 1/A$, $A = [(1+\lambda)\gamma-1]/(1-\lambda)\gamma$.

<u>Proof</u>. For $g(z) \in S$, it is showed by Krzyż and Reade [36] that

$$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \ge \frac{1-r}{1+r}$$

for $|z| = r < \tanh \frac{1}{2}$. This result together with (6.2.4) and (6.3.4) yield the equation giving the radius of starlikeness of $f(z)$ to be

$$H(r) \equiv Ar^3 - 3Ar^2 - 3r + 1 = 0 ,$$

provided $r < \tanh \frac{1}{2}$. Now, $H(0) = 1$, $H(1/3) = -8A/27 < 0$. Hence $H(r)$ has a zero in $(0,1/3)$. It is easy to check that this is the unique zero of $H(r)$ in $(0,1/3)$. Also, $1/3 < \tanh \frac{1}{2}$. The assertion therefore follows and is seen to be sharp for the function

$$f(z) = \frac{1-z}{1+Az} \cdot \frac{z}{(1+z)^2} \quad .$$

Putting $\lambda = 0$, Theorems 1 and 2 of Krzyż and Reade [36] are recovered when we let $\gamma \to \infty$ and $\gamma = 1$ respectively .

6.4 The convexity of certain close-to-convex functions

In this section, we briefly look at the problem of determining the radius of convexity of $f(z) \in \mathsf{N}$ for which

$$\left|\frac{f'(z)}{\lambda f'(z)+(1-\lambda)g'(z)} - \gamma\right| < \gamma \ , \quad \gamma \geq 1 \quad , \quad z \in \Delta \quad ,$$

where $g(z)$ belongs to some subclass of N or S . For such $f(z)$, we can deduce in a similar manner as in the proof of Theorem 6.3.1 that

$$(6.4.1) \qquad 1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zg''(z)}{g'(z)} - \frac{zp'(z)}{p(z)+\mu} \ , \ |z| < 1/A \quad ,$$

where $p(z) \in \mathsf{P}$, $\mu = (1-A)/(1+A)$, $A = [(1+\lambda)\gamma-1]/(1-\lambda)\gamma$. Hence, with some restriction on λ , we may apply (6.2.4) and the known bound for $\mathrm{Re}\{1+zg''(z)/g'(z)\}$ to (6.4.1) to get the equation from which the radius of convexity of $f(z)$ may be obtained. We consider the following six cases, the first three of which are clear from Theorems 6.3.1 , 6.3.2 and 6.3.3.

6.4.1 Theorem. *Let* $f(z) \in N$ *be such that* $f'(z)/[\lambda f'(z)+(1-\lambda)g'(z)] \in P(\gamma)$, *where* $g(z) \in N$ *and* $g'(z) \in P$, $0 \le \lambda < (1+\sqrt{3}+1/2\gamma)/(2+\sqrt{3})$. *Then the radius of convexity of* $f(z)$ *is equal to* σ_1 *as given by Theorem* 6.3.1.

6.4.2 Theorem. *Let* $f(z) \in N$ *be such that* $f'(z)/[\lambda f'(z)+(1-\lambda)g'(z)] \in P(\gamma)$, *where* $g(z) \in N$ *and* $g'(z) \in P_{\frac{1}{2}}$, $0 \le \lambda < (\sqrt{5}+1+1/\gamma)/(\sqrt{5}+3)$. *Then the radius of convexity of* $f(z)$ *is equal to* σ_2 *as given by Theorem* 6.3.2.

6.4.3 Theorem. *Let* $f(z) \in N$ *be such that* $f'(z)/[\lambda f'(z)+(1-\lambda)g'(z)] \in P(\gamma)$, *where* $g(z) \in S_\alpha^c$. *Then the radius of convexity of* $f(z)$ *is equal to* σ_3 *for* $0 \le \lambda < \lambda_0$, σ_3 , λ_0 *being as given in Theorem* 6.3.3.

6.4.4 Theorem. *Let* $f(z) \in N$ *be such that* $f'(z)/[\lambda f'(z)+(1-\lambda)g'(z)] \in P(\gamma)$, *where* $g(z) \in S^*$. *Then the radius of convexity* ρ *of* $f(z)$ *is given by the smallest root in* $(0,1)$ *of the equation*

$$Ar^3 - 5Ar^2 - 5r + 1 = 0$$

for $0 \le \lambda < \lambda_2$, λ_2 *being determined by the condition* $\rho < 1/A$, $A = [(1+\lambda)\gamma-1]/(1-\lambda)\gamma$.

Proof. For $g(z) \in S^*$, we have on $|z| = r < 1$

$$\mathrm{Re}\{1+\frac{zg''(z)}{g'(z)}\} = \mathrm{Re}\{p(z) + \frac{zp'(z)}{p(z)}\} \quad , \quad p(z) \in P$$

$$(6.4.2) \qquad \ge \frac{1-4r+r^2}{1-r^2} \quad \text{, from (1.2.13) and (1.2.3) with } A=1, B=-1.$$

The assertion now follows from (6.4.2) , (6.2.4) and (6.4.1) .

6.4.5 Theorem. *Let* $f(z) \in N$ *be such that* $f'(z)/[\lambda f'(z)+(1-\lambda)g'(z)] \in P(\gamma)$, *where* $g(z) \in S_{\frac{1}{2}}^*$. *Then the radius of convexity of* $f(z)$ *is equal to* σ_4 *for* $0 \le \lambda < \lambda_1$, σ_4 , λ_1 *being as given by Theorem* 6.3.5.

Proof. Theorem 4.1 of Singh and Goel [85] with $\beta = \frac{1}{2}$ gives, for $g(z) \in S^*_{\frac{1}{2}}$,

$$\mathrm{Re}\{1 + \frac{zg''(z)}{g'(z)}\} \geq \frac{1-r}{1+r} \quad , \quad |z| = r < \tfrac{1}{2} \ .$$

This result together with (6.2.4) applied to (6.4.1) yield the same equation as that of Theorem 6.3.5.

The last case we want to consider is $g(z) \in S$. In this case, it is well-known [27, p. 166] that

$$\mathrm{Re}\ \{1 + \frac{zg''(z)}{g'(z)}\} \geq \frac{1-4r+r^2}{1-r^2} \quad , \quad |z| = r < 1 \ ,$$

which is exactly (6.4.2). Thus, in view of Theorem 6.4.4 , we obtain

6.4.6 Theorem. *Let* $f(z) \in N$ *be such that* $f'(z)/[\lambda f'(z)+(1-\lambda)g'(z)] \in P(\gamma)$, *where* $g(z) \in S$. *Then the radius of convexity of* $f(z)$ *is equal to* ρ *for* $0 \leq \lambda < \lambda_2$, ρ , λ_2 *being as given by Theorem* 6.4.4.

All these results are best possible and generalise those by Ratti [70, Theorems 1-6] for the case $\lambda = 0$, $\gamma \to \infty$.

6.5 The radius of convexity for a subclass of regular functions

In Section 6.3 , we have been concerned with the starlikeness of functions $f(z) \in N$ defined by the condition

$$\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)g(z)} - \gamma\right| < \gamma \ , \quad \gamma \geq 1 \quad , \quad z \in \Delta \ ,$$

where $g(z)$ belongs to a subclass of N or S. Putting $\lambda = 0$ and letting $\gamma \to \infty$ in these results, we obtain the radii of starlikeness of corresponding known classes, for example, the class $\mathcal{T}(\alpha)$ of functions $f(z) \in \mathcal{N}$ which satisfy $\mathrm{Re}\{f(z)/g(z)\} > 0$ in Δ, where $g(z) \in S_\alpha^*$. A further question, which arises naturally, is that of determining the radii of convexity of these classes. Sakaguchi [80] proved that the radius of convexity of $\mathcal{T}(0)$ is $5-2\sqrt{6}$. Goel [21] extended Sakaguchi's result to the class $\mathcal{T}(\alpha)$; however his result is again sharp only when $\alpha = 0$. Reade, Ogawa and Sakaguchi [72] obtained the radius of convexity for a subclass of $\mathcal{T}(0)$, namely, the class of functions $f(z) \in \mathcal{N}$ which satisfy $\mathrm{Re}\{f(z)/z\} > 0$ in Δ (see also Section 3.4 of Chapter 3). The method employed in [80], [21] and [72] is based on certain coefficient inequalities for the classes under consideration.

In [72], Reade, Ogawa and Sakaguchi put forward the problem of finding the radius of convexity of the class of functions $f(z) \in \mathcal{N}$ for which $\mathrm{Re}\{f(z)/g(z)\} > 0$ in Δ, where $g(z) \in S^c$ or $g(z) \in S$. This problem was considered by Goel [22] for two special cases when

$$\text{(i)} \quad g(z) = \frac{z}{1-z} \quad \text{and} \quad \text{(ii)} \quad g(z) = \frac{z}{(1-z)^2} .$$

In this section, we determine the radius of convexity for the family of functions $f(z) \in \mathcal{N}$ defined by

$$(6.5.1) \qquad \left|\frac{f(z)}{g(z)} - \gamma\right| < \gamma \ , \quad \gamma \geq 1 \ , \quad z \in \Delta \ ,$$

where $g(z)$ belongs to a subclass of S, namely, the class G of functions $g(z) \in \mathcal{N}$ which satisfy $|g'(z)-1| < 1$ in Δ (see MacGregor [47]). Letting $\gamma \to \infty$ in this result we obtain the radius of convexity for functions $f(z) \in \mathcal{N}$ for which $\mathrm{Re}\{f(z)/g(z)\} > 0$ in Δ, $g(z) \in \mathcal{G}$.

We remark that, for $g(z)$ in an arbitrary subclass and $f(z)$ as given by (6.5.1), it may be deduced that

$$1 + \frac{zf''(z)}{f'(z)} = 1 - \left[\frac{z^2p''(z)}{p(z)+\mu} - \frac{z^2g''(z)}{g(z)}\right]\left[\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu}\right]^{-1} - \frac{2zp'(z)}{p(z)+\mu},$$

where $p(z) \in \mathcal{P}$, $\mu = 1/(2\gamma-1)$. Hence the problem now is to find the sharp upper bounds on $|z| = r$ for

$$\left|\left[\frac{z^2p''(z)}{p(z)+\mu} - \frac{z^2g''(z)}{g(z)}\right]\left[\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu}\right]^{-1}\right|$$

and $|zp'(z)/(p(z)+\mu)|$ and to show that they are attained at the same point. Unfortunately, this can be achieved only in isolated cases (see also Section 3.4).

We need the following lemmas.

6.5.1 <u>Lemma</u>. *If* $w(z) \in \mathcal{B}$, *then* $|w'(z)| \le 1$ *for* $|z| \le \sqrt{2}-1$.

6.5.2 <u>Lemma</u>. *If* $p(z) \in \mathcal{P}$, $\mu > 0$, *then on* $|z| = r < 1$,

$$\left|\frac{zp'(z)}{p(z)+\mu}\right| \le \frac{2r}{(1-r)[1+\mu+(1-\mu)r]}, \tag{6.5.2}$$

$$\left|\frac{z^2p''(z)}{p(z)+\mu}\right| \le \frac{4r^2}{(1-r)^2[1+\mu+(1-\mu)r]}. \tag{6.5.3}$$

A proof for Lemma 6.5.1, which is due to Dieudonné, may be found in Carathéodory [13, p. 19]. Inequalities (6.5.2), (6.5.3) are derived from (3.4.11) and (3.4.14) respectively by putting $t = 1$ in these latter results. Equalities in (6.5.2) and (6.5.3) occur for the function $p(z) = (1+z)/(1-z)$ at $z = r$.

6.5.3 Lemma. *Let* $g(z) \in \mathrm{N}$ *be such that* $|g'(z)-1| < 1$ *in* Δ . *Then on* $|z| = r$,

$$(6.5.4) \qquad \left|\frac{z^2 g''(z)}{g(z)}\right| \le \frac{2r}{2-r} , \quad \textit{for} \quad r < \sqrt{2}-1 ,$$

$$(6.5.5) \qquad \mathrm{Re}\{\frac{zg'(z)}{g(z)}\} \ge \frac{2(1-r)}{2-r} , \quad \textit{for } r < \tfrac{1}{2} .$$

The results are sharp.

Proof. For $g(z)$ as defined, we have $g'(z)-1 = w_1(z)$ for some $w_1(z) \in \mathrm{B}$; hence, in view of Lemma 6.5.1 ,

$$(6.5.6) \qquad |g''(z)| \le 1 \quad , \quad |z| \le \sqrt{2}-1 .$$

Also, from Section 2 of MacGregor [47] ,

$$\left|\frac{g(z)}{z} - 1\right| \le \tfrac{1}{2}|z| \quad ;$$

hence we may write

$$(6.5.7) \qquad g(z) = z + \tfrac{1}{2}zw_2(z) \quad , \quad w_2(z) \in \mathrm{B} \quad , \quad z \in \Delta \quad .$$

This implies

$$(6.5.8) \qquad |g(z)| \ge |z| - \tfrac{1}{2}|z|^2$$

and (6.5.4) now follows from (6.5.6) and (6.5.8) .

From the representation (6.5.7) and Dieudonné's lemma, we get

$$\mathrm{Re}\{\frac{zg'(z)}{g(z)}\} = 1 + \mathrm{Re}\{\frac{zw_2'(z)}{2+w_2(z)}\}$$

$$(6.5.9) \qquad \ge 1 + \mathrm{Re}\{\frac{w_2(z)}{2+w_2(z)}\} - \frac{r^2 - |w_2(z)|^2}{(1-r^2)|2+w_2(z)|} .$$

Put $2+w_2(z) = Re^{i\theta}$ and denote the right-hand side of (6.5.9) by $S(R,\theta)$; then $2-r \le R \le 2+r$ and

$$S(R,\theta) = 2 - \left(\frac{2}{R} + \frac{4}{1-r^2}\right)\cos\theta + \frac{4-r^2}{1-r^2}\cdot\frac{1}{R} + \frac{R}{1-r^2} \quad .$$

Since $\partial S/\partial\theta = \sin\theta T(R)$ and

$$T(R) = \frac{2}{R} + \frac{4}{1-r^2} > 0 \quad ,$$

the minimum of $S(R,\theta)$ occurs when $\theta = 0$ and $R \in [2-r,2+r]$. Now,

$$S(R,0) = \frac{1}{1-r^2}[-2(1+r^2) + (2+r^2)\frac{1}{R} + R] \quad ,$$

which yields $dS(R,0)/dR = 0$ at $R = (2+r^2)^{\frac{1}{2}}$. This point is outside the range of values of R if $(2+r^2)^{\frac{1}{2}} < 2-r$, that is, if $r < \frac{1}{2}$. Thus, for $r < \frac{1}{2}$, the minimum of $S(R,0)$ is attained at the end-point $R = 2-r$, its value being

$$S(2-r,0) = \frac{2(1-r)}{2-r} \quad .$$

The sharpness of both results is easily verified for the function $g_0(z) = z + z^2/2$.

We now prove the main result of this section.

6.5.4 <u>Theorem</u>. *Let* $f(z) \in \mathsf{N}$ *be such that* $f(z)/g(z) \in \mathsf{P}(\gamma)$, *where* $g(z) \in \mathsf{G}$. *Then the radius of convexity of* $f(z)$ *is given by the only root in* $(0,1/4)$ *of the equation*

$$4(1-\gamma)^2r^4 - (1-\gamma)(3+8\gamma)r^3 + 9\gamma r^2 - 2\gamma(7\gamma-1)r + 2\gamma^2 = 0 \quad .$$

Proof. Write $\psi(z) = 1-f(z)/\gamma g(z)$. Then $|\psi(z)| < 1$ in Δ and $\psi(0) = 1-1/\gamma = \psi_0$. Put $w(z) = [\psi(z)-\psi_0]/[1-\psi_0\psi(z)]$; then $w(z) \in \mathcal{B}$ and $\psi(z) = [w(z)+\psi_0]/[1+\psi_0 w(z)]$. In view of this representation and the fact that every $w(z) \in \mathcal{B}$ can be represented by $w(z) = [p(z)-1]/[p(z)+1]$, $p(z) \in \mathcal{P}$, we get

$$(6.5.10) \qquad f(z) = \frac{2\gamma g(z)}{1+(2\gamma-1)p(z)} \quad , \quad z \in \Delta \quad .$$

This yields, putting $\mu = 1/(2\gamma-1)$,

$$\mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\} = 1-\mathrm{Re}\{[\frac{z^2p''(z)}{p(z)+\mu} - \frac{z^2g''(z)}{g(z)}][\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu}]^{-1} - \frac{2zp'(z)}{p(z)+\mu}\}$$

$$\geq 1 - |\frac{z^2p''(z)}{p(z)+\mu} - \frac{z^2g''(z)}{g(z)}||\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu}|^{-1} - |\frac{2zp'(z)}{p(z)+\mu}|$$

$$(6.5.11) \qquad \geq 1 - |\frac{z^2p''(z)}{p(z)+\mu} - \frac{z^2g''(z)}{g(z)}|[\mathrm{Re}\{\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu}\}]^{-1} - |\frac{2zp'(z)}{p(z)+\mu}|$$

provided that $\mathrm{Re}\{zg'(z)/g(z) - zp'(z)/[p(z)+\mu]\} > 0$. From (6.5.5) and (6.5.2) we have

$$(6.5.12) \quad \mathrm{Re}\{\frac{zg'(z)}{g(z)} - \frac{zp'(z)}{p(z)+\mu}\} \geq \frac{2[\mu+1 - 3(\mu+1)r + 3\mu r^2 + (1-\mu)r^3]}{(1-r)(2-r)[1+\mu+(1-\mu)r]} \quad .$$

It is easy to check that the numerator has a single root in (0,1) ; furthermore, this root is located in $(\frac{1}{4},\frac{1}{2})$. Thus the right-hand side of (6.5.12) is positive for $r < \frac{1}{4}$. This fact together with (6.5.2) , (6.5.3) , (6.5.4) and (6.5.12) applied to (6.5.11) will give, for $r < \frac{1}{4}$,

$$\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \geq \frac{G(r)}{\{(1-r)^2[1+\mu+(1-\mu)r] - r(2-r)\}[1+\mu+(1-\mu)r]} \quad ,$$

where

$$G(r) = 2(\mu-1)^2r^4 + (1-\mu)(7\mu+4)r^3 + 9\mu(1+\mu)r^2 - (1+\mu)(5\mu+7)r + (1+\mu)^2 \quad .$$

Now, $G(0) = (1+\mu)^2$, $G(\frac{1}{4}) = (27\mu^2-44\mu-87)/128 < 0$ for $0 < \mu \leq 1$. Thus $G(r)$ has a zero, which is unique, in $(0,\frac{1}{4})$. The proof of the theorem is therefore completed.

The result is sharp for the function

$$f(z) = \frac{(1+z)(z+z^2/2)}{1+(1/\gamma-1)z} \quad .$$

CHAPTER 7

THE β-CONVEXITY OF CERTAIN STARLIKE UNIVALENT FUNCTIONS

7.1 Introduction

In 1955, Bazilevič [9] introduced a class of functions f(z) regular in Δ and defined by the relation

$$(7.1.1)\qquad f(z) = \{ \frac{\beta}{1+\alpha^2} \int_0^z [\, p(\xi)-\alpha i\,]\xi^{-\alpha\beta i/(1+\alpha^2)-1} g(\xi)^{\beta/(1+\alpha^2)} d\xi \,\}^{(1+\alpha i)/\beta},$$

where $p(z) \in P$, $g(z) \in S^*$, α is any real number, $\beta > 0$ and all powers are meant as principal values. He was able to show that each such function is univalent in Δ.

Putting $\alpha = 0$, $p(z) = 1$ in (7.1.1) , we have

$$(7.1.2)\qquad f(z) = \{ \beta\int_0^z g(\xi)^{\beta}\xi^{-1} d\xi \,\}^{1/\beta}.$$

A function satisfying (7.1.2) is known as a Bazilevič function of type β. We denote by $H(\beta)$ the class of all such functions.

From (7.1.2) we obtain, for $f(z) \in H(1/\beta)$, $\beta > 0$,

$$(1-\beta)\frac{zf'(z)}{f(z)} + \beta(1+\frac{zf''(z)}{f'(z)}) = \frac{zg'(z)}{g(z)} .$$

Since $g(z) \in S^*$, it follows that

$$(7.1.3)\qquad \mathrm{Re}\,\{(1-\beta)\frac{zf'(z)}{f(z)} + \beta(1+\frac{zf''(z)}{f'(z)})\} > 0 \ , z \in \Delta .$$

This condition, which is a linear combination of the conditions for convexity and starlikeness, was used by Mocanu [56] to introduce the concept of β-convexity. Thus, a function $f(z) = z+a_2z^2+\ldots$ which is regular and such that $f'(z)f(z)/z \neq 0$ in Δ is said to be β-convex, β real, if it satisfies inequality (7.1.3). The class of all β-convex functions in Δ is denoted by $\mathcal{M}(\beta)$. For simplicity, we shall write

$$J(\beta,f) \equiv (1-\beta)\frac{zf'(z)}{f(z)} + \beta(1+\frac{zf''(z)}{f'(z)}) .$$

Then

$$\mathcal{M}(\beta) = \{ f(z) \in \mathcal{N} ; f'(z)f(z)/z \neq 0 , \mathrm{Re}\{J(\beta,f)\} > 0 , \beta \text{ real}, z \in \Delta \}.$$

It was shown by Miller, Mocanu and Reade [55] and Kulshrestha [37] that $f(z) \in \mathcal{M}(\beta)$ if and only if $f(z) \in \mathcal{H}(1/\beta)$, $\beta > 0$. It is obvious from the definition that $\mathcal{M}(0)\equiv S^*$ and $\mathcal{M}(1)\equiv S^c$. Furthermore, $\mathcal{M}(\beta_2) \subseteq \mathcal{M}(\beta_1)$ whenever $0 < \beta_1 \leq \beta_2 < \infty$; hence all members of $\mathcal{M}(\beta)$ are starlike for $\beta \geq 0$ and convex for $\beta \geq 1$. In fact, a stronger result that $\mathcal{M}(\beta) \subseteq S^*$ for all real β was established by Miller, Mocanu and Reade [54].

For $0 \leq \beta \leq 1$, we have the inclusion relations

$$S^c \subseteq \mathcal{M}(\beta) \subseteq S^* .$$

Indeed, for $f(z) \in S^c$, we have on $|z| = r$

$$\mathrm{Re}\{ 1+ \frac{zf''(z)}{f'(z)} \} \geq \frac{1-r}{1+r} , \quad \text{from (1.2.3) with } A = 1 , B = -1 ,$$

$$\mathrm{Re}\{ \frac{zf'(z)}{f(z)} \} \geq \tfrac{1}{2} (\frac{1-r}{1+r} + 1) = \frac{1}{1+r} , \quad \text{from the fact that } S^c \subseteq S^*_{\frac{1}{2}} .$$

Hence

$$\mathrm{Re}\{ J(\beta,f) \} \geq \frac{1-\beta r}{1+r} ,$$

which is positive for $0 \leq \beta \leq 1$. In other words, $f(z) \in \mathcal{M}(\beta)$ for $0 \leq \beta \leq 1$. And so $S^c \subseteq \mathcal{M}(\beta)$ for $0 \leq \beta \leq 1$.

We have mentioned that $\mathcal{M}(\beta) \subseteq S^*$ for β real. Conversely, we wish to determine the largest number r_β such that each function $f(z) \in S^*$ is β-convex for $|z| < r_\beta$. The complete answer to this question was provided by Mocanu and Reade [57]. The number r_β is called the radius of β-convexity of S^* .

It is of interest to know the radii of β-convexity of certain subclasses of S^*. Al-Amiri [1] gave the radius of β-convexity for S_α^* for the case $\beta \geq 0$; Miller, Mocanu and Reade [55] determined that for $S^*(M)$ also when $\beta \geq 0$. For $\beta \leq 0$, apart from the result due to Mocanu and Reade for S^* mentioned above, Bajpai and Mehrok [8] obtained the radius of β-convexity for S_α^* for the case $0 < \alpha < \frac{1}{2}$, $\beta \leq -1$.

In this chapter, we establish the radius of β-convexity for

(i) $S^*(A, B)$ for $\beta \geq 0$,

(ii) $S^*(A, B)$, $-1 \leq B \leq 0 \leq A \leq 1$, for

$$\beta \leq \begin{cases} -1 , & A + B \geq 0 , \\ -\dfrac{1-A^2}{1+AB} , & A + B \leq 0 . \end{cases}$$

In particular, the radii of β-convexity, for all real β, of the classes S_α^* , $0 \leq \alpha \leq \frac{1}{2}$, and $S^*[\alpha]$ are completely determined. Also, corresponding results for $\sum^*(A, B)$, $\sum_\alpha^*$, $\sum^*[\alpha]$ are given.

Again, by the radius of β-convexity of a class F , we shall mean the number

$$r(\beta,F) = \sup\{ r ; \operatorname{Re}\{ J(\beta,f) \} > 0, |z| < r , f(z) \in F \} .$$

For $f(z) \in S^*(A, B)$, we deduce that

$$J(\beta,f) = p(z) + \beta \frac{zp'(z)}{p(z)} , \quad p(z) \in P(A, B) . \tag{7.1.4}$$

Hence the radius of β-convexity of $S^*(A, B)$ may be obtained by solving the extremal problem

$$\text{(7.1.5)} \qquad \min_{|z|=r<1} \operatorname{Re}\{\, p(z) + \beta \frac{zp'(z)}{p(z)} \,\} , \; \beta \text{ real}$$

over $P(A, B)$. Theorem 1.2.2 with $\alpha = 1$ supplies the solution for the case $\beta \geq 0$. For $\beta \leq 0$, the analysis is much more complicated. We have been able to solve (7.1.5) with $\beta \leq 0$ only when A,B and β are subject to the restrictions stated in part (ii) above.

7.2 <u>The functional $\operatorname{Re}\{\, p(z) + \beta zp'(z)/p(z) \,\}$, $\beta \leq 0$, over $P(A, B)$</u>

7.2.1 <u>Theorem</u>. *If* $p(z) \in P(A, B)$, $-1 \leq B \leq 0 \leq A \leq 1$,

$$\beta \leq \begin{cases} -1 , & A + B \geq 0 , \\ -\dfrac{1-A^2}{1+AB} , & A + B \leq 0 , \end{cases}$$

then on $|z| = r < 1$,

$$\operatorname{Re}\{p(z)+\beta\frac{zp'(z)}{p(z)}\} \geq \begin{cases} \dfrac{1 + (2A+\beta A-\beta B)r + A^2r^2}{(1+Ar)(1+Br)} , & R_8 \geq R_6 , \\[2ex] \dfrac{\beta(A+B)}{A-B} - \dfrac{2\beta}{(A-B)(1-r^2)} [(L_2K_3)^{\frac{1}{2}} - (1-ABr^2)] , & R_6 \geq R_8 , \end{cases}$$

where $R_8 = (L_2/K_3)^{\frac{1}{2}}$, $R_6 = (1+Ar)/(1+Br)$, $L_2 = (1+A)(1-Ar^2)$, $K_3 = (B-A)(1-r^2)/\beta + (1+B)(1-Br^2)$. *The result is sharp.*

<u>Proof</u>. With the same argument as in Theorem 1.2.2, we obtain

$$\text{(7.2.1)} \quad \operatorname{Re}\{p(z)+\beta\frac{zp'(z)}{p(z)}\} \geq \frac{\beta(A+B)}{A-B} + \frac{1}{A-B} \operatorname{Re}\{[A-(1+\beta)B]p(z) - \frac{\beta A}{p(z)}\} + \beta \frac{r^2|Bp(z)-A|^2 - |1-p(z)|^2}{(A-B)(1-r^2)\,|p(z)|} .$$

Put $p(z) = Re^{i\theta}$ and denote the right-hand side of (7.2.1) by $S(R,\theta)$, then

$$(7.2.2)\qquad S(R,\theta) = \frac{\beta(A+B)}{A-B} - \frac{\beta}{A-B}\{[(\frac{B-A+\beta B}{\beta})R + \frac{A}{R} - \frac{2(1-ABr^2)}{1-r^2}]\cos\theta$$

$$+ \frac{1-B^2r^2}{1-r^2}.R + \frac{1-A^2r^2}{1-r^2}\cdot\frac{1}{R}\} .$$

Now

$$\frac{\partial S}{\partial\theta} = T(R)\sin\theta ,$$

where

$$T(R) = \frac{\beta}{A-B}[\frac{B-A+\beta B}{\beta}.R + \frac{A}{R} - \frac{2(1-ABr^2)}{1-r^2}] .$$

Thus

$$\frac{dT}{dR} = -\frac{\beta A}{A-B}(-\frac{B-A+\beta B}{\beta A} + \frac{1}{R^2})$$

$$\geq -\frac{\beta A}{A-B}(-1+\frac{1}{R^2}) \quad , \quad \text{as} \quad \beta \leq -1 .$$

We recall that $R \in [a-d,a+d]$, where a,d are as given by (1.2.2). Since $a-d < 1$, $a+d > 1$, the minimum of $T(R)$ is attained at either $R = a-d$ or $R = a+d$. We shall show that $T(a-d) \geq 0$, $T(a+d) \geq 0$ with some restriction on β. Now,

$$T(a-d) = \frac{B-A+\beta B}{A-B}\cdot\frac{1-Ar}{1-Br} + \frac{\beta A}{A-B}\cdot\frac{1-Br}{1-Ar} - \frac{2\beta}{A-B}\cdot\frac{1-ABr^2}{1-r^2}$$

$$= \frac{N_1 + \beta N_2}{(A-B)(1-r^2)(1-Ar)(1-Br)} ,$$

where

$$N_1 = -(A-B)(1-Ar)^2(1-r^2) ,$$

$$N_2 = B(1-Ar)^2(1-r^2) + A(1-Br)^2(1-r^2) - 2(1-ABr^2)(1-Ar)(1-Br).$$

It can be shown that $A(1-Br)(1-r^2) < 2(1-ABr^2)(1-Ar)$. In fact, put

$$G(A,B) = A(1-Br)(1-r^2) - 2(1-ABr^2)(1-Ar) ;$$

then $\partial G/\partial A = 1-r^2 - Br(1-r^2) + 2r(1+Br) - 4ABr^3 > 0$ as $B \le 0 \le A$. Hence $G(A,B) \le G(1,B)$. Similarly, $G(1,B) \le G(1,-1) = -(1-r)^3 < 0.$ Thus $N_2 < 0$, and so $T(a-d) \ge 0$ if $\beta \le -N_1/N_2$. Now we want to show that

$$-\frac{1-A^2r^2}{1+ABr^2} \le -\frac{N_1}{N_2} ,$$

that is,

$$-\frac{1-A^2r^2}{1+ABr^2} \le \frac{A-B}{B+X} ,$$

where

$$X = A\left(\frac{1-Br}{1-Ar}\right)^2 - 2\,\frac{1-ABr^2}{1-r^2} \cdot \frac{1-Br}{1-Ar} .$$

This inequality is equivalent to

$$X \le -\frac{A(1-B^2r^2)}{1-A^2r^2} ,$$

that is, $(1-A)(1+Ar^2) \ge 0$, which is always true. We also note that

$$\frac{1-A^2r^2}{1+ABr^2} \le \begin{cases} 1 & , \quad A+B \ge 0 , \\ \dfrac{1-A^2}{1+AB} & , \quad A+B \le 0 ; \end{cases}$$

hence the restriction

$$\beta \le \begin{cases} -1 & , A+B \ge 0 , \\ -\dfrac{1-A^2}{1+AB} & , A+B \le 0 , \end{cases}$$

ensures that $T(a-d) \ge 0$. Next, we have

$$T(a+d) = \frac{B-A+\beta B}{A-B} \cdot \frac{1+Ar}{1+Br} + \frac{\beta A}{A-B} \cdot \frac{1+Br}{1+Ar} - \frac{2\beta}{A-B} \cdot \frac{1-ABr^2}{1-r^2}$$

$$= \frac{N}{(A-B)(1+Ar)(1+Br)} ,$$

where

$$N = (B-A)(1+Ar)^2(1-r^2) + \beta[B(1+Ar)^2(1-r^2) + A(1+Br)^2(1-r^2)$$

$$- 2(1-ABr^2)(1+Ar)(1+Br)] \ge 0 \quad \text{if}$$

$$\beta \le \frac{(A-B)(1+Ar)^2(1-r^2)}{B(1+Ar)^2(1-r^2)+A(1+Br)^2(1-r^2)-2(1-ABr^2)(1+Ar)(1+Br)} = \frac{N_3}{N_4} ,$$

as it is clear that $A(1+Br)(1-r^2) < 2(1+Ar)(1-ABr^2)$ for $B \le 0 \le A$. Similarly as above, we can show that

$$-\frac{1-A^2r^2}{1+ABr^2} < \frac{N_3}{N_4} .$$

Hence the same restriction on β will give $T(a+d) \ge 0$. Thus, the minimum of $S(R,\theta)$ on the disc $|p(z)-a| \le d$ is attained when $\theta = 0$ and $R \in [a-d,a+d]$. Setting $\theta = 0$ in (7.2.2) we get

$$S(R,0) = \frac{\beta(A+B)}{A-B} - \frac{\beta}{A-B} \{[\frac{B-A+\beta B}{\beta} + \frac{1-B^2r^2}{1-r^2}] R + (1+A)\frac{1-Ar^2}{1-r^2} \cdot \frac{1}{R} - \frac{2(1-ABr^2)}{1-r^2}\} ,$$

$$\frac{dS(R,0)}{dR} = -\frac{\beta}{A-B} \cdot \frac{1}{R^2}\left[\left(\frac{B-A+\beta B}{\beta} + \frac{1-B^2r^2}{1-r^2}\right)R^2 - (1+A)\frac{1-Ar^2}{1-r^2}\right] .$$

Consequently, the minimum of $S(R,0)$ occurs at $R = R_8$ if $R_8 \in [a-d, a+d]$, its value being

$$S(R_8,0) = \frac{\beta(A+B)}{A-B} - \frac{2\beta}{(A-B)(1-r^2)}\left[(L_2K_3)^{\frac{1}{2}} - (1-ABr^2)\right] .$$

Now we have

$$\frac{(1+A)(1-Ar^2)}{(B-A)(1-r^2)/\beta + (1+B)(1-Br^2)} > \frac{1-Ar^2}{1-Br^2}$$

if and only if $1-Br^2 > (1-r^2)/(-\beta)$. The condition $\beta \le -1$ ensures that this inequality is satisfied . Hence

$$R_8^2 > \frac{1-Ar^2}{1-Br^2} > \frac{1-Ar}{1-Br} > \left(\frac{1-Ar}{1-Br}\right)^2 = (a-d)^2 ,$$

that is, $R_8 > a-d$. However, R_8 is not always less than $a+d$. For the case $R_8 \ge a+d = R_6$, the minimum of $S(R,0)$ occurs at $R = a+d$, its value being

$$S(a+d,0) = \frac{1 + (2A+\beta A-\beta B)r + A^2r^2}{(1+Ar)(1+Br)} .$$

The result is sharp for the function $p_1(z) = (1+Az)/(1+Bz)$ for $R_8 \ge R_6$ and the function $p_5(z) = (1+Aw_4(z))/(1+Bw_4(z))$ for $R_6 \ge R_8$, where $w_4(z) = z(z-c_4)/(1-c_4z)$ with c_4 being determined by $\mathrm{Re}\,\{(1+Aw_4(z))/(1+Bw_4(z))\} = R_8$ at $z = r$.

7.3 The radius of β-convexity of $S^*(A, B)$

For $f(z) \in S^*(A, B)$, we may write

$$(1-\beta)\frac{zf'(z)}{f(z)} + \beta\left(1+\frac{zf''(z)}{f'(z)}\right) = p(z) + \beta\frac{zp'(z)}{p(z)} \quad , \quad p(z) \in P(A, B) \ , \ z \in \Delta \ .$$

Hence an application of Theorem 1.2.2 with $\alpha = 1$ yields

7.3.1 Corollary. *The radius of β-convexity, $\beta \geq 0$, of $S^*(A, B)$ is given by*

(i) *the smallest root r_1 in $(0,1)$ of the equation*

$$A^2r^2 - (2A+\beta A-\beta B)r + 1 = 0 \quad , \quad \textit{for } \beta \geq \beta_1 \ ,$$

(ii) *the smallest root r_2 in $(0,1)$ of the equation*

$$[\beta(A-B) + 4A(1-A)]r^4 + 2[\beta(A-B) + 2(1-A)^2]r^2 + \beta(A-B) + 4A - 4 = 0$$

for $0 \leq \beta \leq \beta_1$, β_1 *being determined from the equation* $r_1 = r_2$.

For the range $\beta \leq 0$, Theorem 7.2.1 can be applied to give

7.3.2 Corollary. *The radius of β-convexity, $\beta \leq 0$, of $S^*(A, B)$ with $-1 \leq B \leq 0 \leq A \leq 1$ is given by*

(i) *the smallest root r_3 in $(0,1)$ of the equation*

$$A^2r^2 + (2A+\beta A-\beta B)r + 1 = 0 \quad , \quad \textit{for } \beta \leq \beta_2 \ ,$$

(ii) *the smallest root r_4 in $(0,1)$ of the equation*

$$[\beta(A-B) + 4A(1+A)]r^4 - 2[2(1+A)^2 - \beta(A-B)]r^2 + \beta(A-B) + 4(1+A) = 0$$

for

$$\beta_2 \leq \beta \leq \begin{cases} -1 & , \quad A+B \geq 0 \ , \\ -\dfrac{1-A^2}{1+AB} & , \quad A+B \leq 0 \ , \end{cases}$$

β_2 *being determined from the equation* $r_3 = r_4$.

Corollary 7.3.2 does not supply a complete result on the

β-convexity of $S^*(A, B)$ with $-1 \le B \le 0 \le A \le 1$ over the whole range $\beta \le 0$. However, for the classes S_α^*, $0 \le \alpha \le \frac{1}{2}$, and $S^*[\alpha]$, the radii of β-convexity may be determined for all $\beta \le 0$.

7.3.3 Theorem. *The radius of β-convexity, $\beta \le 0$, of S_α^*, $0 \le \alpha \le \frac{1}{2}$, is equal to*

$$r_5(\alpha,\beta) = -\{1-2\alpha+\beta(1-\alpha) - [(1-2\alpha+\beta-\alpha\beta)^2 - (1-2\alpha)^2]^{\frac{1}{2}}\}^{-1}, \quad \textit{for} \quad \beta \le \beta_3 ,$$

$$r_6(\alpha,\beta) = \left[\frac{4(1-\alpha)-\beta - 4(\alpha^2+\alpha\beta-\beta)^{\frac{1}{2}}}{\beta + 4(1-2\alpha)}\right]^{\frac{1}{2}}, \quad \textit{for} \quad \beta_3 \le \beta \le 0 ,$$

where β_3 is the smallest root in $(-4,-2(1-\alpha))$ of the equation $r_5(\alpha,\beta) = r_6(\alpha,\beta)$.

Proof. An application of Corollary 7.3.2 with $A = 1-2\alpha$, $B = -1$ yields, for $\beta \le -2(1-\alpha)$, the two equations giving the radius of β-convexity of $f(z)$ to be

$$F_1(r) \equiv (1-2\alpha)^2 r^2 + (2-4\alpha+2\beta-2\alpha\beta)r + 1 = 0 ,$$

$$F_2(r) \equiv [2\beta(1-\alpha)+8(1-2\alpha)(1-\alpha)]r^4 - 2[8(1-\alpha)^2-2\beta(1-\alpha)]r^2 + 2(1-\alpha)\beta + 8(1-\alpha) = 0 .$$

Now, $F_1(0) = 1$, $F_1(1) = 2(1-\alpha)[2(1-\alpha)+\beta] < 0$ if $\beta < -2(1-\alpha)$. Hence $F_1(r)$ has a zero in $(0,1)$ if $\beta < -2(1-\alpha)$. With this restriction on β, the smallest zero in $(0,1)$ of $F_1(r)$ is $r_5(\alpha,\beta)$. Similarly, $F_2(0) = \beta+4 > 0$ if $\beta > -4$, $F_2(1) = 4\beta < 0$. Hence we derive the smallest zero in $(0,1)$ of $F_2(r)$ to be $r_6(\alpha,\beta)$ if $\beta > -4$. The transition point for the two cases may be obtained by solving for $\beta = \beta_3$ the equation $r_5(\alpha,\beta) = r_6(\alpha,\beta)$, where β_3 must lie in the interval $(-4,-2(1-\alpha))$.

With the analysis as that of Theorem 7.2.1, we see that for the range $\beta \ge -2(1-\alpha)$ we might have $T(R) \ge 0$ or $T(R) \le 0$. Hence we could only say that the minimum of $S(R,\theta)$ on any arc R = constant inside the disc $|p(z)-a| \le d$ is reached either when $\theta = 0$ or at the intersections of this arc with the circle $|p(z)-a| = d$. At these points, by writing $p(z) = a+u+iv$, we have

$$r^2|p(z)+1-2\alpha|^2 - |1-p(z)|^2 = (1-r^2)[d^2-(u^2+v^2)] = 0 .$$

Hence, inequality (7.2.1) with $A = 1-2\alpha$, $B = -1$ becomes

$$\mathrm{Re}\{p(z)+ \beta\frac{zp'(z)}{p(z)}\} \ge -\frac{\alpha\beta}{1-\alpha} + \frac{1}{2(1-\alpha)}[(2-2\alpha+\beta)R - \frac{\beta(1-2\alpha)}{R}] \cos\theta \quad .$$

The right-hand side is always positive as $2-2\alpha+\beta \ge 0$ for $\beta \ge -2(1-\alpha)$. This shows that the minimum of $S(R,\theta)$ may vanish only on the diameter $\theta = 0$. Now,

$$S(R,0) = -\frac{\alpha\beta}{1-\alpha} + \frac{1}{2(1-\alpha)}[2(1-\alpha)R - 2\beta(1-\alpha).\frac{1-(1-2\alpha)r^2}{(1-r^2)\ R} - 2\beta\frac{1+(1-2\alpha)r^2}{1-r^2}]$$

so that $dS(R,0)/dR = 0$ at

$$R_9 = [-\beta\frac{1-(1-2\alpha)r^2}{1-r^2}]^{\frac{1}{2}} \quad .$$

We note that R_9 is the same as R_8 if we put $A = 1-2\alpha$, $B = -1$. It is easy to check that $R_9 < a+d$, but R_9 is not always greater than $a-d$. Hence the minimum of $S(R,0)$ occurs either at $R = a-d$ or at $R = R_9$. In the former case,

$$S(a-d,0) = \frac{1-2(1-2\alpha+\beta-\alpha\beta)r+(1-2\alpha)^2r^2}{(1+r)[1-(1-2\alpha)r]}$$

$$= \frac{[1-(1-2\alpha)r]^2 - 2\beta(1-\alpha)}{(1+r)[1-(1-2\alpha)r]} \quad \ge \quad \frac{1-(1-2\alpha)r}{1+r}$$

which is always positive for $0 < r < 1$. For $R = R_9$, $S(R_9,0)$ vanishes at $r = r_6(\alpha,\beta)$ which is the smallest zero in (0,1) of $F_2(r)$. Hence the radius of β-convexity of f(z) is $r_6(\alpha,\beta)$ for $\beta \geq -2(1-\alpha)$.

As a special case of Corollary 7.3.1 and Theorem 7.3.3 we have

7.3.4 <u>Corollary</u>. *The radius of β-convexity, β real, of* $S^*_{\frac{1}{2}}$ *is*

$$r_\beta = \begin{cases} \{[-(\beta+2) + 2(2\beta+1)^{\frac{1}{2}}]/\beta\}^{\frac{1}{2}} & , \quad 0 \leq \beta \leq \sqrt{2}+1 , \\ 1/\beta & , \quad \beta \geq \sqrt{2}+1 , \\ -1/\beta & , \quad \beta \leq -\sqrt{2}-1 , \\ \{[2-\beta-2(1-2\beta)^{\frac{1}{2}}]/\beta\}^{\frac{1}{2}} & , \quad -\sqrt{2}-1 \leq \beta \leq 0 . \end{cases}$$

We recall that Padmanabhan's class $S^*[\alpha]$ of starlike univalent functions is defined as

$$S^*[\alpha] = \{f(z) \in N ; \left|\frac{zf'(z)}{f(z)} -1\right| / \left|\frac{zf'(z)}{f(z)} + 1\right| < \alpha , \ 0 < \alpha \leq 1, \ z \in \Delta\} .$$

For this class, we prove

7.3.5 <u>Theorem</u>. *The radius of β-convexity, $\beta \leq 0$, of $S^*[\alpha]$ is equal to*

$$r_7(\alpha,\beta) = -[1+\beta+(2\beta+\beta^2)^{\frac{1}{2}}]/\alpha \quad , \quad \textit{for} \ \beta \leq \beta_4 \ ,$$

$$r_8(\alpha,\beta) = \left\{\frac{(1+\alpha)^2-\alpha\beta-(1+\alpha)[(1-\alpha)^2-4\alpha\beta]^{\frac{1}{2}}}{\alpha\beta+2\alpha(1+\alpha)}\right\}^{\frac{1}{2}} \quad , \quad \textit{for} \ \beta_4 \leq \beta \leq 0 \ ,$$

where β_4 is the smallest root in $(-2(1+\alpha)/\alpha, -(1+\alpha)^2/2\alpha)$ of the equation $r_7(\alpha,\beta) = r_8(\alpha,\beta)$.

<u>Proof</u>. Since $S^*[\alpha] \equiv S^*(\alpha,-\alpha)$, Corollary 7.3.2 with $A = \alpha$, $B = -\alpha$

gives, for $\beta \le -1$, the two equations which determine the radius of β-convexity of $S^*[\alpha]$ to be

$$G_1(r) \equiv \alpha^2 r^2 + 2\alpha(1+\beta)r + 1 = 0 \ ,$$

$$G_2(r) \equiv [\alpha\beta+2\alpha(1+\alpha)]r^4 - 2[(1+\alpha)^2-\alpha\beta]r^2 + \alpha\beta+2(1+\alpha) = 0 \ .$$

Now, $G_1(0) = 1$, $G_1(1) = (1+\alpha)^2 + 2\alpha\beta < 0$ if $\beta < -(1+\alpha)^2/2\alpha$. Hence $G_1(r)$ has a zero in $(0,1)$ if $\beta < -(1+\alpha)^2/2\alpha$. It is clear that under this condition on β , $r_7(\alpha,\beta)$ is the only zero in $(0,1)$ of $G_1(r)$. For $G_2(r)$, we have that $G_2(0) = \alpha\beta+2(1+\alpha) > 0$ if $\beta > -2(1+\alpha)/\alpha$ and $G_2(1) = 8\alpha\beta < 0$. Hence $G_2(r)$ has a zero in $(0,1)$ if $\beta > -2(1+\alpha)/\alpha$. Under this condition on β, the smallest zero in $(0,1)$ of $G_2(r)$ is $r_8(\alpha,\beta)$. The transition point for the two cases is therefore obtained by solving for $\beta = \beta_4$ the equation $r_7(\alpha,\beta) = r_8(\alpha,\beta)$, where β_4 must lie in the interval $(-2(1+\alpha)/\alpha, -(1+\alpha)^2/2\alpha)$.

The same argument as in the proof of Theorem 7.3.3 shows that for $-1 \le \beta \le 0$, the minimum of $S(R,\theta)$ on any arc R = constant inside the disc $|p(z)-a| \le d$ is reached either when $\theta = 0$ or at the intersections of this arc with the circle $|p(z)-a| = d$. At these points, by writing $p(z) = a+u+iv$, we find

$$\alpha^2 r^2 |p(z)+1|^2 - |1-p(z)|^2 = (1-\alpha^2 r^2)[d^2-(u^2+v^2)] = 0 \ .$$

Hence, inequality (7.2.1) with $A = \alpha$, $B = -\alpha$ becomes

$$\operatorname{Re}\{p(z)+\beta\frac{zp'(z)}{p(z)}\} \ge \frac{1}{2\alpha}[\alpha(2+\beta)R - \frac{\alpha\beta}{R}]\cos\theta$$

which is always positive for $-1 \le \beta \le 0$. Consequently, the minimum of $S(R,\theta)$ may vanish only on $\theta = 0$. Now,

$$S(R,0) = -\frac{\beta}{2\alpha}\{[\frac{1-\alpha^2r^2}{1-r^2} - \frac{\alpha(2+\beta)}{\beta}]R + (\frac{1-\alpha^2r^2}{1-r^2}+\alpha)\frac{1}{R} - \frac{2(1+\alpha^2r^2)}{1-r^2}\} .$$

Hence $dS(R,0)/dR = 0$ at

$$R_0 = [\frac{(1+\alpha)(1-\alpha r^2)}{(1-\alpha)(1+\alpha r^2)-2\alpha(1-r^2)/\beta}]^{\frac{1}{2}} .$$

With $a-d = (1-\alpha r)/(1+\alpha r)$ and $a+d = (1+\alpha r)/(1-\alpha r)$ for the class under consideration , we find $R_0 < a+d$ if and only if

$$(1+\alpha)(1-\alpha r^2)(1-\alpha r)^2 < [(1-\alpha)(1+\alpha r^2) - 2\alpha(1-r^2)/\beta](1+\alpha r)^2 .$$

Since $-1 \le \beta \le 0$, this condition holds if

$$(1+\alpha)(1-\alpha r^2)(1-\alpha r)^2 < [(1-\alpha)(1+\alpha r^2) + 2\alpha(1-r^2)](1+\alpha r)^2 ,$$

or equivalently, $\alpha r^2 < 1$, which is always true for $0 < \alpha \le 1$, $0 < r < 1$. Hence $R_0 < a+d$; however R_0 is not always greater than a-d . At $R = a-d$,

$$S(a-d,0) = -\frac{\beta}{2\alpha}[\frac{(1-\alpha)(1+\alpha r^2)-2\alpha(1-r^2)/\beta}{1-r^2}\cdot\frac{1-\alpha r}{1+\alpha r} + \frac{(1+\alpha)(1-\alpha r^2)}{1-r^2}\cdot\frac{1+\alpha r}{1-\alpha r} - 2\frac{1+\alpha^2r^2}{1-r^2}]$$

$$= \frac{N}{2\alpha(1-r^2)(1-\alpha^2r^2)} ,$$

where

$$N = -\beta(1-\alpha)(1+\alpha r^2)(1-\alpha r)^2+2\alpha(1-r^2)(1-\alpha r)^2-\beta(1+\alpha)(1-\alpha r^2)(1+\alpha r)^2+2\beta(1-\alpha^4r^4)$$

$$\ge -\beta[(1-\alpha)(1+\alpha r^2)(1-\alpha r)^2+(1+\alpha)(1-\alpha r^2)(1+\alpha r)^2-2(1-\alpha^4r^4)]$$

$$= -4\beta\alpha^2r(1-r^2)$$

which is always positive for $0 < r < 1$. Hence the minimum of $S(R,0)$ occurs at $R = R_0$ for $-1 \le \beta \le 0$. Since $S(R_0,0)$ vanishes at $r = r_8(\alpha,\beta)$

which is the smallest zero in (0,1) of $G_2(r)$, the radius of β-convexity of $S^*[\alpha]$ is $r_8(\alpha,\beta)$ for $-1 \leq \beta \leq 0$. The proof of the theorem is thus completed.

7.3.6 Remark. As noted previously, $S^c \subseteq S^*_{\frac{1}{2}}$; in other words, for $f(z) \in N$, the inequality $\mathrm{Re}\{1+zf''(z)/f'(z)\} > 0$ in Δ implies the inequality $\mathrm{Re}\{zf'(z)/f(z)\} > \frac{1}{2}$ in Δ . Recently, McLaughlin [53] derived further inequalities which connect the quantities $\mathrm{Re}\{zf'(z)/f(z)\}$ and $\mathrm{Re}\{1+zf''(z)/f'(z)\}$. In particular, this author showed that if $\mathrm{Re}\{zf'(z)/f(z)\} > 0$ in Δ , that is, $f(z) \in S^*$, then for each $k > 1$, there exists a radius $r(k) > 0$ such that the relation

$$(7.3.1) \qquad \mathrm{Re}\{1+zf''(z)/f'(z)\} \leq k\, \mathrm{Re}\{zf'(z)/f(z)\}$$

holds in the disc $|z| \leq r(k)$. For $k = 2$, $r(k) = [(k-1)/(k+1)]^{\frac{1}{2}}$ and, for $k \geq 3$, the radius $r(k)$ satisfies the condition

$$1-(k-2)^{-\frac{1}{2}} \leq r(k)^2 \leq (k-1)/(k+1) .$$

In view of the bounds for the cases $k = 2$ and $k \geq 3$, McLaughlin raised the question whether inequality (7.3.1) holds for $|z| \leq [(k-1)/(k+1)]^{\frac{1}{2}}$ for all $k > 1$. The answer is negative and is given in the next corollary, which is a direct consequence of Theorem 7.3.3.

7.3.7. Corollary. *Let* $f(z) \in S^*$ *and* $k > 1$. *Then*

$$\mathrm{Re}\{1+\frac{zf''(z)}{f'(z)}\} \leq k\, \mathrm{Re}\{\frac{zf'(z)}{f(z)}\}$$

for $|z| < r_k$, *where*

$$r_k = \begin{cases} \dfrac{2-k-(3-2k)^{\frac{1}{2}}}{k-1} , & 1 < k \le 4/3 , \\ \left[\dfrac{2(k-1)^{\frac{1}{2}}-1}{2(k+1)^{\frac{1}{2}}+1}\right]^{\frac{1}{2}} , & k \ge 4/3 . \end{cases}$$

The result is sharp.

Proof. For $f(z) \in S^*$, $zf'(z)/f(z) = p(z)$, $z \in \Delta$, for some $p(z) \in P$. Hence the above inequality may be rewritten as

$$\operatorname{Re}\{(k-1)p(z) - \frac{zp'(z)}{p(z)}\} \ge 0 \quad ,$$

or equivalently,

$$\operatorname{Re}\{p(z) + \beta\frac{zp'(z)}{p(z)}\} \ge 0 \ , \quad \beta = -(k-1)^{-1} < 0 \ .$$

The result now follows from Theorem 7.3.3 with $\alpha = 0$ and is sharp for

$$f(z) = \frac{z}{(1-z)^2} \quad , \qquad \text{for} \quad 1 < k \le 4/3 \ ,$$

$$f(z) = \exp \int_0^z \frac{1+w_0(\xi)}{1-w_0(\xi)} \, d\xi \quad , \quad \text{for} \quad k \ge 4/3 \ ,$$

where $w_0(z) = z(z-c_0)/(1-c_0z)$ with c_0 such that $\operatorname{Re}\{[1+w_0(z)]/[1-w_0(z)]\} = (k-1)^{-\frac{1}{2}}$ at $z = r$.

7.4 The radius of β-convexity of $\sum^*(A, B)$

Let us now look at the problem of determining the radius of β-convexity of the class $\sum^*(A, B)$ of meromorphic starlike functions.

For a function $f(z) \in \sum^*(A, B)$, we may write

$$(7.4.1) \qquad J(\beta,f) = p(z) - \beta\frac{zp'(z)}{p(z)} \quad , \quad p(z) \in P(A, B) \ , \quad z \in \Delta \ .$$

Hence the radius of β-convexity of $\sum^*(A, B)$ is given by the solution of the extremal problem

$$(7.4.2) \qquad \min_{|z|=r<1} \operatorname{Re}\{p(z)-\beta\frac{zp'(z)}{p(z)}\} \quad , \beta \text{ real}$$

over $P(A, B)$. In view of (7.1.4) , (7.1.5) , (7.4.1) and (7.4.2) , we conclude that the radius of β-convexity of $\sum^*(A, B)$ may be deduced from that of $S^*(A, B)$ and vice versa. Consequently, from Corollaries 7.3.1 and 7.3.2 we get

7.4.1 Corollary. *The radius of β-convexity, $\beta \le 0$, of $\sum^*(A, B)$ is given by*

(i) *the smallest root in* (0,1) *of the equation*

$$A^2r^2 - (2A-\beta A+\beta B)r + 1 = 0 \ , \quad \textit{for} \quad \beta \le -\beta_1 \ ,$$

(ii) *the smallest root in* (0,1) *of the equation*

$$[4A(1-A)-\beta(A-B)]r^4+2[2(1-A)^2-\beta(A-B)]r^2-\beta(A-B)+4A-4 = 0 \ ,$$

for $-\beta_1 \le \beta \le 0$, β_1 *being as defined in Corollary* 7.3.1.

7.4.2 Corollary. *The radius of β-convexity, $\beta \ge 0$, of $\sum^*(A, B)$ with* $-1 \le B \le 0 \le A \le 1$ *is given by*

(i) *the smallest root in* (0,1) *of the equation*

$$A^2r^2 + (2A-\beta A+\beta B)r + 1 = 0 \ , \quad \textit{for} \quad -\beta_2 \le \beta \ ,$$

(ii) *the smallest root in* (0,1) *of the equation*

$$[4A(1+A)-\beta(A-B)]r^4-2[2(1+A)^2+\beta(A-B)]r^2-\beta(A-B)+4(1+A) = 0 \ ,$$

for

$$-\beta_2 \ge \beta \ge \begin{cases} 1 & , \ A+B \ge 0 \ , \\ \dfrac{1-A^2}{1+AB} & , \ A+B \le 0 \ , \end{cases}$$

β_2 *being as defined in Corollary* 7.3.2.

In particular, for the classes $\sum_{\alpha}^{*}$ and $\sum^{*}[\alpha]$ (see Section 5.3 of Chapter 5), Theorems 7.3.3 and 7.3.5 imply immediately

7.4.3 <u>Corollary</u>. *The radius of β-convexity, $\beta \geq 0$, of $\sum_{\alpha}^{*}$, $0 \leq \alpha \leq \frac{1}{2}$, is equal to*

$$r_{\beta} = \begin{cases} r_5(\alpha,-\beta) & , \quad -\beta_3 \leq \beta \ , \\ r_6(\alpha,-\beta) & , \quad 0 \leq \beta \leq -\beta_3 \ , \end{cases}$$

where $r_5(\alpha,\beta)$, $r_6(\alpha,\beta)$, β_3 *are as given in Theorem* 7.3.3.

7.4.4 <u>Corollary</u>. *The radius of β-convexity, $\beta \geq 0$, of $\sum^{*}[\alpha]$ is equal to*

$$r_{\beta} = \begin{cases} r_7(\alpha,-\beta) & , \quad -\beta_4 \leq \beta \ , \\ r_8(\alpha,-\beta) & , \quad 0 \leq \beta \leq -\beta_4 \ , \end{cases}$$

where $r_7(\alpha,\beta)$, $r_8(\alpha,\beta)$, β_4 *are as given in Theorem* 7.3.5.

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